THREE ESSAYS IN OPERATIONS MANAGEMENT

by

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Dedicated to my parents, my wife, and teachers.

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PREFACE

This dissertation was produced in accordance with guidelines which permit the inclusion as part of the dissertation the text of an original paper or papers submitted for publication. The dissertation must still conform to all other requirements explained in the "Guide for the Preparation of Master's Theses and Doctoral Dissertations at The University of Texas at Dallas." It must include a comprehensive abstract, a full introduction and literature review, and a final overall conclusion. Additional material (procedural and design data as well as descriptions of equipment) must be provided in sufficient detail to allow a clear and precise judgment to be made of the importance and originality of the research reported.

It is acceptable for this dissertation to include as chapters authentic copies of papers already published, provided these meet type size, margin and legibility requirements. In such cases, connecting texts which provide logical bridges between different manuscripts are mandatory. Where the student is not the sole author of a manuscript, the student is required to make an explicit statement in the introductory material to that manuscript describing the student's contribution to the work and acknowledging the contribution of the other author(s). The signatures of the Supervising Committee which precede all other material in the dissertation attest to the accuracy of this statement.

THREE ESSAYS IN OPERATIONS MANAGEMENT

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This dissertation addresses three important problems in operations management.

The aim of the first problem is to highlight – and demonstrate via specific examples – the need for algorithmic results for some fundamental set-based notions on which search in social networks is expected to be prevalent. To this end, we argue that the concepts of *an influential set* and *a central set* that highlight, respectively, the specific role and specific location of a set are likely to be useful in practice. We formulate two specific search problems: the Elite Group Problem (EGP) and the Portal Problem (PP), that represent these two concepts and provide a variety of algorithmic results.

In the second paper, we introduce two new multi-period models – designed specifically to capture the operations of a medium-size Depository Institution (DI) – that emerge from the DI's objective to minimize the total cost incurred in managing the inventory of cash over a finite planning horizon. The Basic Model (BM) captures the DI's mode of operations if it

chooses not to locally reuse cash and, instead, incur the cross-shipping penalty. The Reuse Model (RM) represents the DI's actions when it locally recirculates cash. We show that the *Value of Local Reuse* for a DI, measured as the percentage cost saving between the optimal solutions of BM and RM, is substantial.

The third problem investigates the concept of *Industrial Symbiosis*, a resource-sharing strategy among co-located firms to engage traditionally separate industries in a collective approach that involves physical exchanges of materials, water, energy and by-products. Inspired by a real-world example of a paper-sugar symbiotic complex, we study the impact on a firm's operational decisions from implementing an industrial symbiotic system. Our focus is on understanding how the introduction of competition and nature of competition will affect the firm's willingness to implement the symbiotic system.

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INTRODUCTION

1.1 Structural Search and Optimization in Social Networks

The explosive growth in the variety and size of social networks has focused attention on searching these networks for useful structures. Like the internet or the telephone network, the ability to efficiently search large social networks will play an important role in the extent of their use by individuals and organizations alike. However, unlike these domains, search on social networks is likely to involve measures that require a *set* of individuals to collectively satisfy some skill requirement or be tightly related to each other via some underlying social property of interest.

The aim of this paper is to highlight – and demonstrate via specific examples – the need for algorithmic results for some fundamental set-based notions on which search in social networks is expected to be prevalent. To this end, we argue that the concepts of *an influential set* and *a central set* that highlight, respectively, the specific role and specific location of a set are likely to be useful in practice. We formulate two specific search problems: the Elite Group Problem (EGP) and the Portal Problem (PP), that represent these two concepts and provide a variety of algorithmic results. We first demonstrate the relevance of EGP and PP across a variety of social networks reported in the literature. For simple networks (e.g., structured trees and bipartite graphs, cycles, paths, etc.), we show that an optimal solution to both EGP and PP is easy to obtain. Next, we show that EGP is polynomially solvable on a general graph while PP is strongly NP-hard. Motivated by practical considerations, we also discuss (1) a size-constrained variant of EGP together with its penalty-based relaxation and (2) the solution of PP on balanced and full d-trees and general trees.

1.2 Value of Local Cash Reuse: Inventory Models for Medium-Size Depository Institutions under the New Federal Reserve Policy

The effective local reuse of physical cash by Depository Institutions (DIs) is the primary goal of the new cash recirculation policy of the Federal Reserve (Fed) of the United States. These guidelines, implemented since July 2007, encourage the reuse of cash by (1) penalizing a DI for the practice of *cross-shipping*: near-simultaneous deposit of used cash to – and withdrawal of fit cash from – the Fed and (2) offering a *custodial inventory* program that enables a DI to transfer fit cash to the Fed's books, but physically hold it within the DI's secured facility. The effective management of the inventory of cash under these new guidelines is both a challenging and important issue for DIs.

We introduce two new multi-period models – designed specifically to capture the operations of a medium-size DI – that emerge from the DI's objective to minimize the total cost incurred in managing the inventory of cash over a finite planning horizon. The Basic Model (BM) captures the DI's mode of operations if it chooses not to locally reuse cash and, instead, incur the cross-shipping penalty. Using two important structural properties, we provide a polynomial-time dynamic programming algorithm for BM. The Reuse Model (RM) represents the DI's actions when it locally recirculates cash. We first prove the hardness of RM and then develop an integer programming formulation. A comprehensive test bed – based on our interaction with a leading secure-logistics provider – helps us to develop several useful insights into the relative impacts of the DI-specific parameters and the Fed's cross-shipping fee on the effective management of cash. In particular, we show that the *Value of Local Reuse* for a DI, measured as the percentage cost saving between the optimal solutions of BM and RM, is substantial and analyze the forces that influence the volume of cross-shipping. We also develop a rolling-horizon procedure to adapt the optimal solutions of BM and RM for obtaining near-optimal real-time solutions in the presence of a modest amount of uncertainty. Finally, we provide a comparative analysis of a DI's decisions under the Fed's mechanism and those under a socially-optimal mechanism.

1.3 Process Innovation via an Industrial Symbiotic System: The Impact of Competition on the Willingness to Implement

Industrial Symbiosis is defined as a resource-sharing strategy among co-located firms to engage traditionally separate industries in a collective approach that involves physical exchanges of materials, water, energy and by-products. Inspired by a real-world example of a paper-sugar symbiotic complex, we study the impact on a firm's operational decisions from implementing an industrial symbiotic system. The two products manufactured by the firm are symbiotically connected, in the sense that the waste from the manufacture of one product is used as a raw material for the second product, and vice-versa. We characterize the firm's operational optimal/equilibrium decisions for its two products – both in the presence and absence of a symbiotic system – under monopoly as well as under competition. The focus of our analysis is on understanding the behavior of the firm's "willingness" to implement.

Our models capture the supply-side (e.g., incurring a fixed cost and changes in variable production costs) as well as the demand-side influence of implementing the symbiotic system. The implementation enables the firm to label its two products as "green" (environment-friendly). The consumers who have a relatively higher valuation for the green variants are further differentiated, based on their loyalty to the green variants. We consider two settings of Cournot competition – depending on whether or not competitors can produce the green variants – in the presence of multiple types of consumers and multiple variants. The difference in the firm's total profits before and after implementation is used as a metric to capture its willingness to implement the system. Three dominating forces that influence the firm's strategic decision to implement the system are (1) the proportion of the green consumers in the market, (2) consumers' appreciation for the green variants, and (3) changes in the variable production costs after implementation. When the firm's variable production costs remain similar before and after implementation, then (1) if the proportion of green consumers is high, then the arrival of competitors who only produce regular variants encourages the implementation of a symbiotic system, (2) if consumers' appreciation of the green variants is low, then the firm's willingness to implement is more (than in a monopoly) under competition from firms who produce both variants. Also of interest are situations when competition incentivizes the firm to implement and both the firm and consumers benefit from the implementation. We identify two such scenarios (one for each type of competition), when the fixed cost of implementing the system is modest.

CHAPTER 2

STRUCTURAL SEARCH AND OPTIMIZATION IN SOCIAL NETWORKS *

2.1 Synopsis

A social network represents a social structure as a set of definite relationships between the members – entities or groups – of a social system. In its most commonly used representation, a social network can be viewed as a network of nodes (individuals, organizations, web pages, etc.) related to one another using edges (friendship, commercial transactions, url links, etc.). Over the years, social networks have been used to analyze social phenomena in a wide variety of domains, including sociology, epidemiology, social psychology, economics, anthropology, history, and human geography (Scott 2000, Wasserman and Faust 1994, Brandes and Erlebach 2005). Often in social network analysis the interest is to explain individual or group behavior in the context of the larger social structure in which the individual or group is situated.

More recently, "social networking sites" such as Facebook (http://www.facebook.com) and Myspace (http://www.myspace.com) have proliferated on the internet and help users connect based on a wide range of interests and practices. While some sites support the maintenance of pre-existing social networks, others help strangers connect based on their shared interests

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and/or activities. Some sites cater to diverse audiences while others attract people based on some shared identity (Boyd and Ellison 2007). Typically, the participants (players) of the network derive some utility from the network, for example, to find each other for exchanging ideas, solving problems, companionship, and so on.

It should be clear that, like any other network-based phenomenon such as the telephone or the internet, the ability of the individual or group to derive value depends on the ability to search the network for contacts. For example, searching the telephone network is facilitated by a phone directory, browsing the internet requires a browser and a search engine, and so on. Many researchers believe that the advent of the web browser and search engine was most influential to the explosive growth of the internet. By analogy, it can be proposed that the utility of social networks to individuals and organizations will also depend on the ability to search the networks of interest for useful structures. For example, a participant in Facebook may want to discuss a topic of interest and may need to call upon a selected subset of friends to join the discussion. In the Open Source community, individual developers form a social network by virtue of having worked on common projects. In such a community, a developer or a firm may want to create a project-team of members with certain specialized skills and access to resources.

Searching a social network often creates search problems that are different from those encountered in other network phenomena like the web or the telephone network. In the web, the typical nature of search is to provide the user with a set of web sites that match based upon a list of search terms. There is usually no requirement that the web sites returned by the search engine satisfy some complex relationship to one another, other than, of course, the trivial relationship that they must all match (to varying degrees) with the list of search terms.

On the other hand, search problems in a social network can be more complex. In particular, the search results may often need to satisfy a *set* measure. For example, in extracting a project-team from a larger network, it may be important that the set of developers that are returned collectively satisfy some skill requirements, but, in addition, are tightly related to one another by virtue of having worked on common projects. With the improvement in computing technology, the data and the tools needed to identify the network of interest are readily available. From a technical perspective, when the results of a search need to meet (or exceed) a specified set measure (specifically, a non-additive measure), the search often becomes combinatorial in nature. Search problems in social networks therefore provide a challenging ground for researchers interested in applying graph-theoretic, algorithmic methods to the area. Our interest in this study stems from the new problems and opportunities that are likely to arise for the use of graph-theoretic methods to solve interesting search problems in social networks.

The remainder of the paper is organized as follows. In Section 2.2, we argue that two set-based notions – influential sets and central sets – are likely to provide a fundamental structural basis for important search problems arising in a variety of practical social networks, and introduce two optimization problems – EGP and PP – corresponding to these two notions. Section 2.3 investigates the complexities of these problems on several special graphs as well as on general graphs. Section 2.4 concludes the paper and provides directions for future research.

2.2 The Notions of Influential Sets and Central Sets

Given the significance of search in social networks and, consequently, the need for efficient algorithms, an important question arises naturally: What are some fundamental set-based notions on which search in social networks is expected to be prevalent? Traditionally, in social network analysis, two fundamental properties of individual members – their *location* and their *role* in the network – have proven to be fundamental. This is natural, since these two properties provide insights into the groupings and interactions in the network. Accordingly, for individual members of a social network, network centrality measures, including *Degree Centrality*, *Closeness Centrality*, and *Betweenness Centrality*, have been heavily investigated and used (see, e.g., Freeman 1979; Brandes and Erlebach 2005, Chapters 3-5). For set-based search too, structures and measures that highlight the specific role or specific location of a set are likely to be the most useful in practice. The need and use of such set-based measures has already been documented in recent studies. For example, the notions of group (or set) betweeness and group degree centralities are discussed in Chapter 4 of Carrington et al. (2005) and in Everett and Borgatti (1999).

The motivation to study the role played by members in a network has to do with understanding the influence a member can potentially cast over other members in the network. Such notions of influence exerted by a single member can intuitively be extended to the influence a *set* of members can potentially exert over the rest of the group. A set of influential members may be useful to identify for a variety of reasons, often having to do with wanting to promote an idea, product, or message to other members of the network. For example, a firm may wish to advertise a new product or service and use an influential group of members to help in this cause (Hill et al. 2006). Similarly, a welfare organization may want to disseminate ideas of social importance within a community of interacting members and use an influential set of members for spreading the message in an effective and timely manner. Another purpose to study influential groups is often to identify a set of members who possess specialized knowledge or information pertaining to a specific domain, namely, the key *experts* in the group. For example, it may be important to identify a set of expert oncologists for devising an informed, yet balanced plan of action to treat a difficult case. Here, a *set* of experts may be especially relevant to consult to eliminate or reduce bias as well as to surface fresh perspectives that can aid in problem solving.

The motivation to study the location of a member (or a set of members) is subtly different from that of examining member roles. Location is essentially a topological characteristic that has to do with a member or a set of members acting to facilitate contact between other interacting members of the network. A centrally located member is *well connected*, or, in other words, has better access to other members by virtue of acting as a conduit that allows exchanges and flows of information or ideas in the network. A central location does not necessarily imply influence, neither does an influential member necessarily need to be centrally located. Indeed, recent research in Reality Mining (Pentland 2004, Greene 2008, Hesseldahl 2008) and interaction within social networks reveals significant distinctions between these two concepts. For example, managers who may be influential within a business organization usually do not play a central role in the routing of communications between teams (Gloor et al. 2007, Thompson 2008). The players central for communication could, instead, be less influential employees. The question arises: what property does location convey that is useful to a problem solver? One benefit of identifying centrally located members is that it provides one with an understanding of the paths that are heavily used in the network so that sufficient resources can be made available at these locations to avoid communication bottlenecks from occurring. An interesting variant is one where the problem solver may *want* to thwart communication: the activities of a terrorist group may be significantly impaired by striking at locations or members that are central to the flow of communication within the network (Erickson 1981).

The twin notions of influence and centrality admit a variety of interpretations, depending on the context of the social network under consideration. Accordingly, there can be several meaningful measures to evaluate "good" influential and central sets. For example, to measure the centrality of a set of vertices, the classical measure of *betweenness centrality* of a single vertex has been extended to *group betweenness centrality* (Everett and Borgatti 1999) and *co-betweenness centrality* (Kolaczyk et al. 2009). Along this theme, we propose two specific measures: one for an influential set and the other for a central set. We now describe two optimization problems that correspond to these two measures, discuss their origins, and provide examples of social networks where these problems are relevant.

2.2.1 The Elite Group Problem (EGP) and The Size-Constrained Elite Group Problem (SCEGP)

Technical Definition

<u>Instance</u>: n players; an "influence" social network represented by a directed graph $G(V, A)$, $|V| = n$, in which the nodes represent the players and the set of arcs represent pairwise

influences pertaining to a social property: a directed arc (i, j) indicates that i is influenced by j. For SCEGP, a positive integer $k \leq n$ is also given.

Solution of EGP: A set $W \subseteq V$ such that there does not exist a directed arc $(i, j) \in A$ with $i \in W$, $j \notin W$.

Solution of SCEGP: Same as EGP, with the additional requirement that $|W| \leq k$.

Objective Function: Maximize the total number of directed arcs, γ_W, incident on any node in W from nodes in $V \backslash W$. More precisely, the *score* γ_W is defined as follows: $\gamma_W = \sum_{i \notin W, j \in W} a_{ij}$, where $a_{ij} = 1$, if $(i, j) \in A$; 0 otherwise.

Note that in a graph $G(V, A)$, there is at least one feasible solution for EGP, namely the complete set of nodes V, with score $\gamma_V = 0$. Also, observe that adding more nodes to an elite group does not necessarily increase the number of directed arcs into the group. If a node, say j, is added to an elite group W, then to obtain $\gamma_{W \cup \{j\}}$, we (i) add to γ_W the number of arcs from $V \backslash \{W \cup \{j\}\}$ to j and (ii) subtract the number of arcs from node j to the nodes in W. Thus, depending on this tradeoff, $\gamma_{W \cup \{j\}}$ can be either larger or smaller or the same as γ_W.

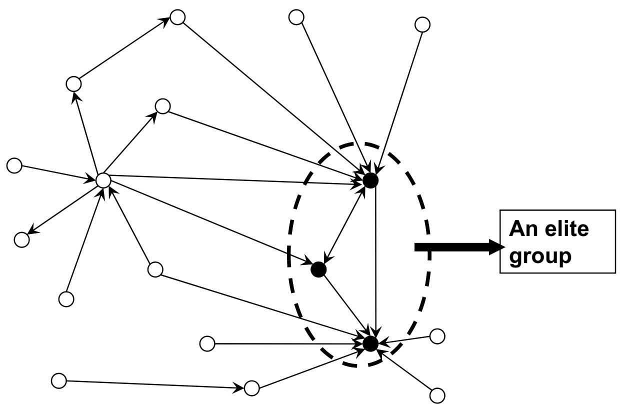

Figure 2.1. An “Influence” Network and an Elite Group.

Origin and Applications

The notion of an "elite" group originated from efforts to examine and understand social behavior within a close-knit community. In the 1980s, Sociologist Li Fan analyzed the giving (and receiving) of gifts between the residents of a Mongolian town (Wellman et al. 2001), and found that one (elite) block of residents received gifts from the others but only exchanged gifts among each other. Thus, as a set, this group of residents only received gifts from the other members of the town.

Another example of the notion of an elite group occurs in the analysis of the advice-seeking behavior of the members of a school, reported in Hawe and Ghali (2008). Here, the social network revealed that, together, the Principal, the Vice-Principal, and some key technical staff, form a group with the properties that (1) most of the other staff members seek advice from one or more members of this group and (2) the members of the group typically seek advice only from (one or more) members within the group. Thus, to influence opinion within the community in general, it may be beneficial to first convince this group of individuals.

In the context of social network analysis, the members of an elite group can be regarded as key players or opinion leaders. For instance, if a member of a community is often consulted by other members on (say) health issues, then she can be regarded as a key player (an "elite" member) in the opinion-seeking network of that community (Borgatti 2006). Another example is the co-sponsorship network in the United States Senate (Fowler 2006). In this network, the prominent senators typically receive a significant amount of co-sponsorship. Thus, the set of these prominent senators constitute an (approximate) elite group.

2.2.2 The Portal Problem (PP) and The Exact-Size Portal Problem (ESPP)

Technical Definition

INSTANCE: n players; a connected, undirected graph $G = (V, E)$, $|V| = n$, in which the nodes represent the players and edges represent the pairwise connections between the players; a positive integer $k \leq n$.

SOLUTION: For PP, a set $Q \subseteq V$ such that $|Q| \leq k$. For ESPP, a set $Q \subseteq V$ such that $|Q| = k$.

OBJECTIVE FUNCTION: Maximize $r(Q)$, defined as follows:

$$r(Q) = \frac{BC(Q)}{\binom{n-|Q|}{2}} \text{ and } BC(Q) = \sum_{s \notin Q, t \notin Q, s \neq t} \frac{\sigma_{st}(Q)}{\sigma_{st}}$$

where σ_{st} is the total number of shortest paths from node s to node t; $s, t \in V \backslash Q$, $s \neq t$, and $\sigma_{st}(Q)$ is the number of shortest paths from node s to node t which have at least one node in set Q as an internal node.

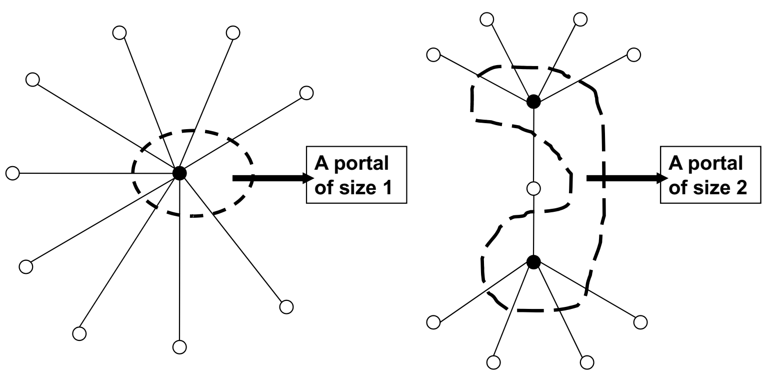

Figure 2.2. Optimal Portals in Two Simple Networks.

Previous Work and Applications

PP is a natural extension of the popular Betweenness Centrality (BC) measure (Freeman

1979, Scott 2000) for individual nodes (members) of a social network; for $k = 1$, an optimal solution to PP is a node with the highest BC. Everett and Borgatti (1999) extend the notion of BC to groups, and illustrate the measure on a few examples. For a *given* set of nodes Q, Puzis et al. (2007) provide a polynomial-time algorithm to compute the non-normalized measure $BC(Q)$ (referred to as "Group Betweenness Centrality"). They prove that the problem of obtaining a set with the highest Group Betweenness Centrality (i.e., $\max_{Q \subseteq V} BC(Q)$) is NP-hard and also propose a simple heuristic. Note that the behavior of our normalized measure $r(Q)$ can be fundamentally different from that of $BC(Q)$. In general, the set which maximizes the Group Betweenness Centrality may not necessarily be an optimal solution of our model, and vice-versa. Puzis et al. (2007) discuss an interesting application of a network of computers in which a limited number of virus-cleaning devices need to be placed at a subset of nodes (computers) to prevent the spread of viruses. To maximize the utility of the devices, it is beneficial to place them at the nodes of a portal of an appropriate size. Another interesting application where a portal may need to be identified is in a disease-outbreak network. For example, Klovdahl et al. (2001) describe a TB-outbreak network and motivate the need to identify the critical members in this network to control the spread of the disease. Everett and Borgatti (1999) discuss the interaction network of animals (monkeys) and use the notion of a portal to determine a socially central set of animals.

2.3 Algorithmic Analysis

We now analyze EGP and PP. For a search problem, a basic question is that of its computational complexity. For simple networks, an optimal solution to both problems is easy to obtain. For EGP, we first illustrate this and then identify a structural property of an elite group that can help in reducing the size of the underlying graph. Then, we show that EGP is polynomially solvable for a general network. Next, motivated by practical considerations, we introduce a size-constrained version of EGP together with its penalty-based relaxation and show that both are strongly NP-hard. For PP, we first show that PP is strongly NP-hard on a general graph. We then consider several special graphs on which PP is polynomially solvable. Finally, we discuss a heuristic for general trees.

2.3.1 The Elite Group Problem (EGP)

Given a directed graph $G(V, A)$, recall that an elite group is a set $W \subseteq V$ such that there does not exist any directed arc $(i, j) \in A$ with $i \in W$, $j \notin W$. The objective of EGP is to maximize the total number (or score), γ_W, of directed arcs incident on the nodes in W. For some simple networks, it may be straightforward to prove the optimality of a specific elite group. Rooted up- and down-trees are especially useful networks to study because they represent hierarchically organized structures, e.g., reporting relationships in a department, natural taxonomies, etc. (Cross and Parker 2004).

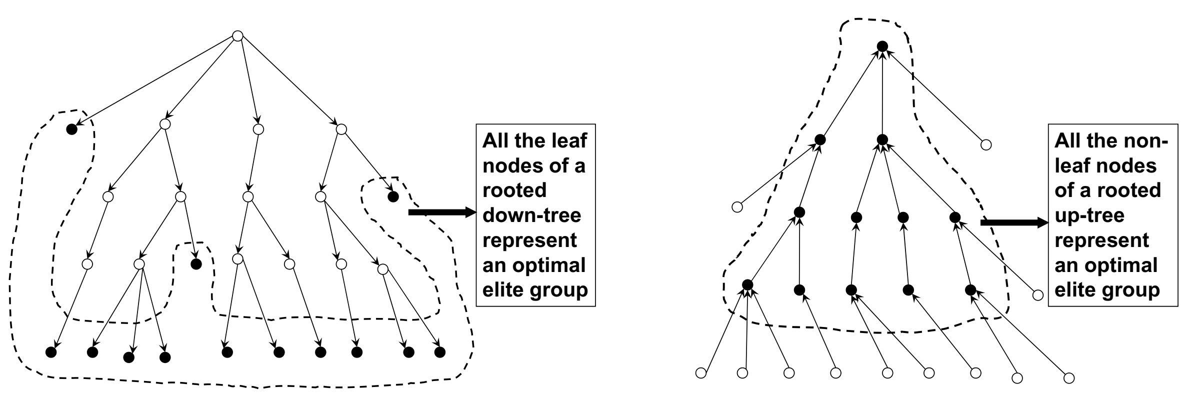

Figure 2.3. Optimal Elite Group for a Rooted Down-Tree and a Rooted Up-Tree.

Observation 2.3.1 *If the graph G is a rooted down-tree (i.e., each node in G, except the root, has a unique predecessor and all arcs in G are directed downwards from the root to the leaf nodes. See Figure 2.3), then the elite group W^* consisting of all the leaf nodes of G is an optimal elite group. If the graph G is a rooted up-tree (i.e., each node in G, except the root, has a unique successor and all arcs in G are directed upwards from the leaf nodes towards the root. See Figure 2.3), then the elite group W^* consisting of all non-leaf nodes of G is an optimal elite group.*

Proof: First, note that the root is not included in an optimal elite group; for otherwise, each node of G is in the elite group and the score is 0, which is clearly a non-optimal solution for any non-trivial rooted down-tree G. Consider an optimal elite group W which contains a non-leaf node t such that the unique predecessor of t is not in W. Note that all descendants of t are also in W. Let $n_t \geq 1$ be the number of direct descendants of t in G. Then, removing t from W results in a feasible elite group $W' = W - \{t\}$ with score $\gamma_{W'} = \gamma_W + (n_t - 1) \geq \gamma_W$. Continuing, we can similarly remove all non-leaf nodes from W without decreasing the score to obtain an elite group consisting only of leaf nodes. Thus, there exists an optimal elite group W^* consisting only of leaf nodes. Finally, note that W^* must contain *all* leaf nodes. This follows since including a leaf node strictly increases the score of an elite group.

The proof for a rooted up-tree is similar. □

Our next result helps us "shrink" the strongly connected components (e.g. directed cycles) in G to single nodes in our search for an elite group. We will use this result later in the proof of Theorem 2.3.3.

Observation 2.3.2 *If G contains a strongly connected component, and at least one node on this component belongs to an elite group W (respectively, the complement $\overline{W} = V \backslash W$), then all the other nodes on the component must belong to W (respectively, $\overline{W}$).*

Proof: Consider a strongly connected component C which includes nodes $v_1, v_2, \ldots, v_n$. Suppose, without loss of generality, $v_1 \in W$. Since there is a directed path from v_1 to v_2, node v_2 must belong to W as well. Continuing this argument, nodes $v_3, v_4, \cdots, v_n$ must belong to W. Similarly, if, say, $v_1 \in \overline{W}$, then $\{v_2, v_3, \ldots, v_n\} \subseteq \overline{W}$. For otherwise, if $v_j \in W$ for some $j \in \{2, 3, \ldots, n\}$, then $v_1 \in W$. The result follows. $\square$

Note that there are many polynomial algorithms to find a strongly connected component (if one exists) in a graph. If G contains a strongly connected component C, then, by using Observation 2.3.2, we can shrink C into a single node. An arc in the original graph between a node $v \in V \backslash C$ and a node of C is represented in the *shrunk graph* as between v and the (shrunk) node representing the component. Thus, in the shrunk graph, we use a separate new arc to represent each arc between a node in $V \backslash C$ and a node of C in the original graph. Therefore, in general, the shrunk graph becomes a multigraph since there may be parallel arcs between two nodes. We can continue this type of shrinking until there is no non-trivial strongly connected component in the modified shrunk graph. Consequently, we can assume without loss of generality that the network is a Directed Acyclic Graph (DAG). Therefore, EGP translates into finding a sink set of maximum indegree in a DAG. The following result follows immediately from Observation 2.3.2.

Lemma 2.3.1 *There is a one-to-one correspondence between elite groups in G and elite groups in the shrunk graph: For every elite group W in G, we can get one in the shrunk* *graph with the same score by taking the nodes of the shrunk graph corresponding to all the strongly connected components in W. Conversely, for every elite group W' in the shrunk graph, we can get one in G with the same score by taking the nodes in all the strongly connected components corresponding to the nodes in W'.*

Next, we show that EGP is polynomially solvable.

Theorem 2.3.2 *The EGP is polynomially solvable.*

Proof: For $j \in V$, define $\pi_j \in \{0,1\}$ as follows:

$$\pi_j = \begin{cases} 1, & \text{node } j \text{ belongs to the elite group } W; \\ 0, & \text{otherwise.} \end{cases}$$

Then, an integer programming (IP) formulation for EGP is as follows:

$$\begin{aligned} \max \sum_{(i,j)\in A} & (\pi_j - \pi_i) \\ s.t. \qquad \pi_i - \pi_j & \leq 0, \qquad \forall\, (i,j) \in A \\ \pi_i & \in \{0,1\}, \qquad \forall\, i \in V \end{aligned}$$

For a directed arc (i,j), if node i is in W (i.e., $\pi_i = 1$), then node j must also be in W (i.e., $\pi_j = 1$). Otherwise, if $\pi_i = 0$, then $\pi_j \in \{0,1\}$. This is enforced by the first constraint. In the objective function, $(\pi_j - \pi_i)$ represents the contribution of arc (i,j) to γ_W: if nodes i and j are both in W or both in $V \backslash W$ (i.e., $\pi_i - \pi_j = 0$), then the contribution is 0. If node i is in $V \backslash W$ and node j is in W (i.e., $\pi_i = 0$, $\pi_j = 1$), then the contribution is 1. The constraints of the IP can be written as $A\pi \leq 0$, where A is the node-arc incidence matrix of G and $\pi \in \{0,1\}^{|V|}$. It is well-known that the node-arc incidence matrix of a *directed* graph is totally unimodular (see, e.g., Hoffman and Kruskal 1956, Nemhauser and Wolsey 1988).

Thus, the linear programming relaxation of the above IP results in an integer optimum. The result follows. ■

Note that the shrinking of strongly connected components (Lemma 2.3.1) maintains the total unimodularity of the constraint matrix of the IP above. Thus, the size of a network containing strongly connected components can be reduced before formulating the EGP. The objective function of the IP above is to maximize the number of arcs directed into the elite group; instead, if we change the objective function to maximize a weighted linear combination of arcs directed into the elite group, the modified problem remains polynomially solvable.

Remark 1: (penalizing the size of the elite group) Note that EGP does not impose any constraint on the cardinality (i.e., the number of nodes) of the elite group. In Section 2.3.1, we consider a hard constraint on the cardinality. An alternative is to impose a "soft" constraint by imposing a penalty $p > 0$ on the cardinality. In this case, the objective function in the IP above changes to $\max \sum_{(i,j)\in A} (\pi_j - \pi_i) - p \sum_{i\in V} \pi_i$. Since this modified objective is linear and the constraint matrix is totally unimodular, the modified problem remains polynomially solvable.

Size-Constrained Elite Group Problem (SCEGP)

Typically, the purpose of identifying an elite group is to use the members of this group to effectively influence the other members of the social network. Thus, for practicability in managing this subsequent task, the size of an elite group may need to be restricted. Motivated by this requirement, Theorem 2.3.3 discusses the complexity of the *Size-Constrained*

Elite Group Problem (SCEGP), defined as follows: Given a positive integer $k \leq n$, find an optimal elite group $W \subseteq V$ with $|W| \leq k$.

Theorem 2.3.3 *The decision problem corresponding to SCEGP is strongly NP-Complete.*

Proof: The strongly NP-Complete problem which we use in our reduction is the Balanced Biclique Problem (Garey and Johnson 1979), defined as follows.

Balanced Biclique Problem (BBP)

Instance. An undirected Bipartite Graph $G = (U \cup V, E)$, with $|U| = |V| = n$. A positive integer $k \leq n$.

Solution. An induced subgraph $G_1 \subseteq G$ such that $G_1 = (U_1 \cup V_1, E_1), U_1 \subseteq U, V_1 \subseteq V$, $|U_1| = |V_1| = k, E_1 \subseteq E$, and $u_1 \in U_1, v_1 \in V_1$ implies that $\{u_1, v_1\} \in E_1$. The *size* of the biclique is $2k$.

Given an arbitrary instance of BBP specified by G, we construct an instance of SCEGP on a related graph G'. The construction of G' is done in two steps. First, we obtain G^c, the bipartite complement graph of G. Then, we add two additional node sets O and S, extend each node in U into a directed cycle, and give directions to all edges to get G'. We now explain our construction and illustrate with an example of G in Figure 2.4:

Step 1 . Get G^c, the bipartite complement graph of G (see Figure 2.4).

Step 2 . We add two node sets O and S consisting, respectively, of n^3 and n^2 nodes. The nodes of O (resp., S) form a directed cycle. There is a directed arc from each node $o_i \in O$ to each node in U. There is a directed arc from each node in V to each node $s_i \in S$. Let $m = n + n^2$. Next, we extend each node $u_i \in U$ into a length m directed cycle

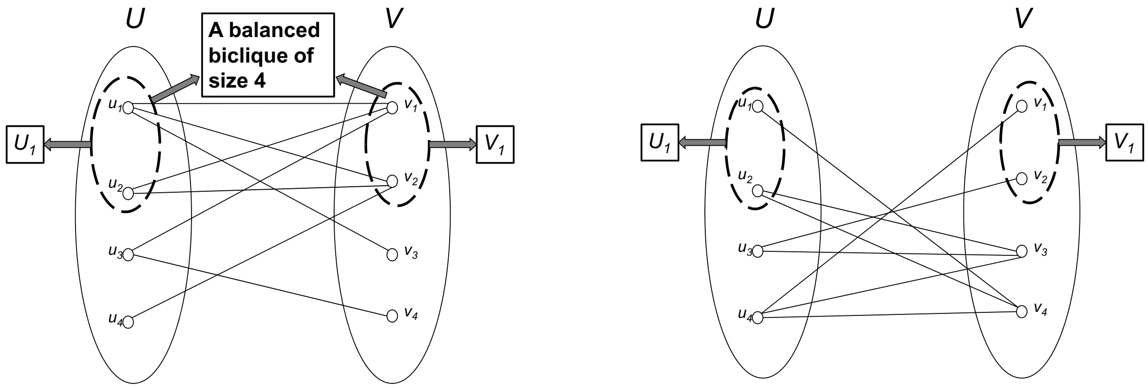

Figure 2.4. A Bipartite Graph G with a Balanced Biclique, and Its Bipartite Complement Graph G^c, Which is Used in the Proof of Theorem 2.3.3.

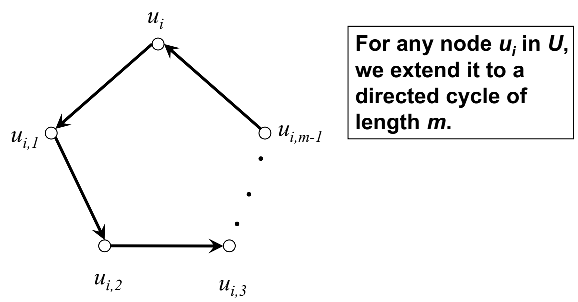

Figure 2.5. The Widget, a Directed Cycle with Length m, Used in the Proof of Theorem 2.3.3.

C_i by adding $m-1$ additional nodes $(u_{i,1}, u_{i,2}, \cdots, u_{i,m-1})$ (see Figure 2.5). Let $U' = \{u_i, u_{i,1}, u_{i,2}, \cdots, u_{i,m-1} | u_i \in U, i = 1, 2, \cdots, n\}$. The edges between O and U' are directed from O to U', those between U' and V are directed from U' to V, and those between V and S are directed from V to S. The construction of G' is now complete (see Figure 2.6). Let $N = O \cup U' \cup V \cup S$. On G', consider the following decision question for $SCEGP$:

DECISION QUESTION: Let $t = km + (n-k) + n^2$ and $D = kn^3 + kn^2$. Does there exist an elite group W in G' such that $|W| \leq t$, and $\gamma_W \geq D$?

Note that the construction of the decision problem from the given instance of the BBP is polynomially bounded. That is, the total number of nodes in G' is bounded by a polynomial in n, as is the time necessary to construct a description of the input of the decision problem.

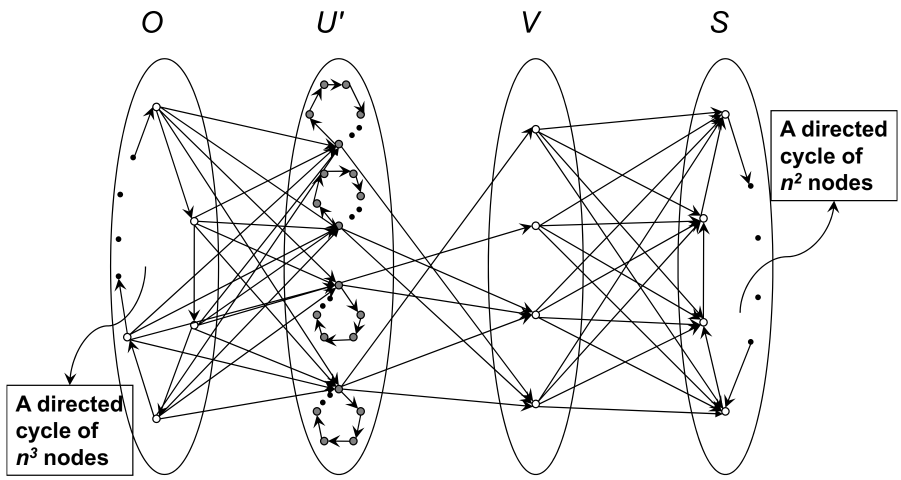

Figure 2.6. The Constructed Graph G' for $SCEGP$.

The decision problem is clearly in class NP. We now show that the decision question has an affirmative answer if and only if the original graph G contains a balanced biclique of size $2k$ (i.e, $|U_1| = |V_1| = k$).

$\Longrightarrow$ Suppose $U_1 \cup V_1$ is a balanced biclique of size $2k$ in G. Let $U_2 = U \backslash U_1, V_2 = V \backslash V_1$. In G', let $U_1' = \{C_i | u_i \in U_1\}$, $U_2' = \{C_i | u_i \in U_2\}$, $W = U_1' \cup V_2 \cup S$, $\overline{W} = O \cup U_2' \cup V_1$ (see Figure 2.7). We now show that the set W is an elite group that provides an affirmative answer to the decision question.

First we need to prove the set W is a valid elite group in G', i.e., there is no arc from W to $\overline{W}$. Since $U_1 \cup V_1$ is a biclique of G, then there is no arc from U_1' to V_1 in G'. Since G is bipartite, there is no arc between U_1' and U_2'. Also, by construction, there is no arc from U_1' to O. Thus, there is no arc from U_1' to $\overline{W}$. Similarly, there is no arc from V_2 to $\overline{W}$ and from S to $\overline{W}$. Thus, W is a valid elite group.

Next, observe that $|W| = |U_1'| + |V_2| + |S| = km + (n-k) + n^2 = t$. Finally, note that γ_W is the number of arcs from $\overline{W}$ to W. The number of arcs from O to U_1' (respectively, V_1 to

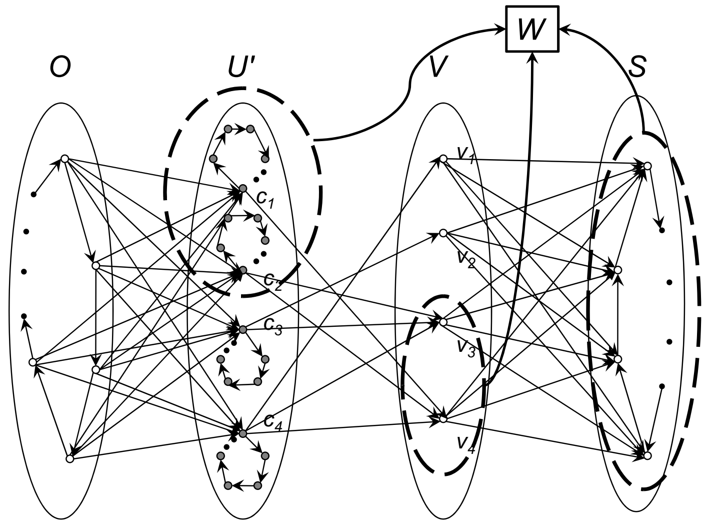

Figure 2.7. Graph G' with Elite Group Set W.

S) is kn^3 (respectively, kn^2). Also, the number of arcs from U'_2 to V_2 is nonnegative. Thus, $\gamma_W \geq kn^3 + kn^2 = D$. The result follows.

$\Longleftarrow$ Suppose W is an elite group in G' with $|W| \leq t$ and $\gamma_W \geq D$. Let $\overline{W} = N \backslash W$. The following claims characterize the set W.

Claim 2.3.1 *In G', the nodes in C_i either all belong to W or all belong to $\overline{W}$. Similarly, the nodes in S (respectively, O) either all belong to W or all belong to $\overline{W}$.*

Proof: The nodes in C_i (respectively, S, O) form a directed cycle. The result follows from Observation 2.3.2. □

Claim 2.3.2 *Each node in O must belong to $\overline{W}$. Similarly, each node in S must belong to W.*

Proof: Suppose a node in O belongs to W. Then, from Claim 2.3.1, each node in O belongs to W. Also, from the definition of elite group, each node in U' must belong to W. Consequently $|W| \geq |O| + |U'| = n^3 + nm$. Since $n \geq 2$ and $n \geq k$, we have $n^3 > n^2 + n >$

$n^2 + n - k$ and $nm \geq km$. So $n^3 + nm > (n^2 + n - k) + km$, which implies $|W| > t$. This contradicts the assumption that $|W| \leq t$. Thus, each node in O must belong to $\overline{W}$.

Suppose a node in S belongs to $\overline{W}$. Then, from Claim 2.3.1, each node in S belongs to $\overline{W}$. Also, each node in V must belong to $\overline{W}$. As shown above, each node in O is in $\overline{W}$. Thus, only a subset $Q' \subseteq U'$ can belong to W. Let $Q = U \cap W$. Note that $|W| = |Q'| = m|Q|$. Since $m = n + n^2$ and $|W| \leq t = n^2 + km + n - k = km + m - k = (k+1)m - k$, we have $|W| = m|Q| \leq (k+1)m - k$, so $|Q| \leq k$. Thus $\gamma_W = n^3|Q| \leq n^3 k < kn^3 + kn^2 = D$, which contradicts the assumption that $\gamma_W \geq D$. Thus, each node in S belongs to W. □

As a consequence of Claim 2.3.2, we have $W = U_1' \cup V_2 \cup S$ and $\overline{W} = O \cup U_2' \cup V_1$. Let $U_1 = \{u_i | C_i \in U_1'\}$.

Claim 2.3.3 $|U_1| = k$.

Proof: We first show that $|U_1| \leq k$. Suppose $|U_1| \geq k+1$, then $|W| \geq |U_1'| = |U_1|m \geq (k+1)m = km+m$. Since $m = n+n^2 > (n-k)+n^2$, we have $|W| \geq km+m > km+n-k+n^2 = t$, which contradicts the assumption that $|W| \leq t$. Thus, $|U_1| \leq k$.

Next, we show that $|U_1| \geq k$. Suppose $|U_1| \leq k-1$. Let $|V_1| = h$. Then, $|V_2| = |V| - |V_1| = n - h$. Recall that γ_W is the number of arcs from $\overline{W}$ to W.

The number of arcs from O to U_1' (resp., from V_1 to S and from U_2' to V_2) is $n^3|U_1| \leq n^3(k-1)$ (resp., hn^2 and $\leq n|V_2| = n(n-h)$). Thus $\gamma_W \leq n^3(k-1) + hn^2 + n(n-h) = kn^3 - n^3 + n^2 + h(n^2 - n)$. Since $n^2 - n > 0$ and $0 \leq h \leq n$, $(n^2 - n)h$ reaches its maximum when $h = n$. Thus $kn^3 - n^3 + n^2 + h(n^2-n) \leq kn^3 - n^3 + n^2 + n(n^2 - n) = kn^3 < kn^3 + kn^2 = D$. Thus, $\gamma_W < D$, contradicting the assumption that $\gamma_W \geq D$. Thus, $|U_1| \geq k$. The result follows. □

Claim 2.3.4 $|V_1| \geq k$.

Proof: Note that $|W| = |U_1'| + |V_2| + |S| = km + |V_2| + n^2 \leq t = km + (n-k) + n^2$. Thus, $|V_2| \leq n - k$. Since $|V_1| = n - |V_2|$, we have $|V_1| \geq k$. □

Note that $U_1' \subseteq W$, $V_1 \subseteq \overline{W}$. Then, from the definition of an elite group, there is no arc from U_1' to V_1 in G'. Since G' is the bipartite complement graph of G, there is an edge between each node in U_1 and each node in V_1 in G. Since $|U_1| = k, |V_1| \geq k$, there exists at least one balanced biclique of size $2k$ in G. This concludes the proof of Theorem 2.3.3. ■

Relaxing the Structure of the Elite Group

Due to the combined requirements of cardinality and structure, the size-constrained elite group problem (SCEGP) of Section 2.3.1 is not guaranteed to always have a non-trivial feasible solution. However, in practice, we may sometimes prefer to identify a non-empty group of players who can influence a large number of players outside the group, but are also influenced by a few outsiders. To enable such solutions, we relax the constraint that forbids directed arcs from nodes of an elite group W to $V \backslash W$. Instead, for each arc, we impose a penalty $p > 0$ that is specified as part of the input. The objective function is the number of incoming arcs into W minus p times the number of arcs coming out of W. We refer to this problem as the *Size-Constrained Elite Group Problem with Penalties (SCEGPP).*

Technical Definition

<u>Instance</u>: n players; an "influence" social network represented by a directed graph $G(V, A)$, $|V| = n$, in which the nodes represent the players and the set of arcs represent pairwise

influences pertaining to a social property: a directed arc (i, j) indicates that i is influenced by j. A positive integer $k \leq n$. A positive number p.

<u>SOLUTION</u>: A set $W \subseteq V$ such that $|W| \leq k$.

<u>OBJECTIVE FUNCTION</u>: Maximize γ_W, the number of arcs from $V \backslash W$ to W minus p times the number of arcs from W to $V \backslash W$. That is, $\gamma_W = \sum_{i \notin W, j \in W} a_{ij} - p \sum_{i \in W, j \notin W} a_{ij}$, where $a_{ij} = 1$, if $(i, j) \in A$; 0 otherwise.

IP Formulation

For $j \in V$, we define $\pi_j \in \{0, 1\}$ as follows:

$$\pi_j = \begin{cases} 1, & \text{node } j \text{ belongs to the elite group } W; \\ 0, & \text{otherwise.} \end{cases}$$

For $(i, j) \in A$, we define $x_{ij} \in \{0, 1\}$ and $y_{ij} \in \{0, 1\}$ as follows:

$$x_{ij} = \begin{cases} 1, & \text{arc } (i, j) \text{ is from } V \backslash W \text{ to } W; \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{arc } (i, j) \text{ is from } W \text{ to } V \backslash W; \\ 0, & \text{otherwise.} \end{cases}$$

An IP formulation for SCEGPP is as follows:

$$\begin{array}{rrcll} & \max \gamma_W & = & \sum_{(i,j) \in A} (x_{ij} - p y_{ij}) & \\ s.t. & \pi_i - \pi_j & = & y_{ij} - x_{ij}, & \forall \, (i, j) \in A \\ & y_{ij} + x_{ij} & \leq & 1, & \forall \, (i, j) \in A \\ & \sum_{i \in V} \pi_i & \leq & k, & \end{array}$$

$$x_{ij}, y_{ij} \in \{0,1\}, \qquad \forall\,(i,j) \in A$$

$$\pi_i \in \{0,1\}, \qquad \forall\, i \in V$$

If node $i \in W$ and node $j \in V\backslash W$ (i.e., $\pi_i = 1$, $\pi_j = 0$), then the first constraint enforces that $y_{ij} = 1$, $x_{ij} = 0$. Similarly, if node $i \in V\backslash W$ and node $j \in W$ (i.e., $\pi_i = 0$, $\pi_j = 1$), then $y_{ij} = 0$, $x_{ij} = 1$. For $\pi_i = \pi_j = 0$ or $\pi_i = \pi_j = 1$, we have $y_{ij} - x_{ij} = \pi_i - \pi_j = 0$ and, hence, $y_{ij} = 0, x_{ij} = 0$ due to the second constraint. Thus, in the objective function, $(x_{ij} - py_{ij})$ represents the contribution of arc (i,j) to γ_W: (i) if arc (i,j) is from $V\backslash W$ to W, then the contribution is 1, (ii) if arc (i,j) is from W to $V\backslash W$, then the contribution is $-p$, (iii) if arc (i,j) is from W to W (or from $V\backslash W$ to $V\backslash W$), then the contribution is 0. The following theorem discusses the computational complexity of SCEGPP.

Theorem 2.3.4 *The decision problem corresponding to SCEGPP is strongly NP-Complete.*

Proof: We again use BBP (Section 2.3.1) in our reduction. The notation is as defined in Section 2.3.1. The construction of the graph G' is exactly the same as in the proof of Theorem 2.3.3. The decision question, however, is different.

DECISION QUESTION: Let $t = km + (n-k) + n^2$, $p = n^4 + 3n^3 + 3n^2$, and $D = kn^3 + kn^2$. Does there exist an elite group W in G' such that $|W| \leq t$, and $\gamma_W \geq D$?

The decision problem is clearly in class NP. It is easy to see that the decision question has an affirmative answer if and only if the original graph G contains a balanced biclique of size $2k$ (i.e, $|U_1| = |V_1| = k$).

$\Longrightarrow$ This part is exactly as same as in the proof of Theorem 2.3.3.

$\Longleftarrow$ Suppose W is an elite group in G' with $|W| \leq t$ and $\gamma_W \geq D$.

Claim 2.3.5 *In G', there does not exist any arc from W to $\overline{W}$.*

Proof: If there exists at least one arc from W to $\overline{W}$, then $\gamma_W \leq \sum_{(i,j)\in A} x_{ij} - p$. Since $\sum_{(i,j)\in A} x_{ij} \leq |G'| \leq p$, we have $\gamma_W \leq 0$, which contradicts $\gamma_W \geq D$. □

With Claim 2.3.5, the remainder of the argument is the same as in the proof of Theorem 2.3.3.

■

2.3.2 The Portal Problem (PP)

Given a connected, undirected graph $G(V, E)$ and a positive integer k, recall from Section 2.2 that an optimal portal is a set $Q \subseteq V, |Q| \leq k$ such that $r(Q)$ is maximized.

As mentioned earlier, a portal is a natural extension to a set-based measure of the notion of Betweenness Centrality (BC) for a single node. For $k = 1$, PP reduces to the well-known Betweenness Centrality Problem, which is polynomially solvable (Everett and Borgatti 1999). Thus, PP is polynomially solvable when $k = 1$. However, for higher values of k, finding an optimal solution is often a challenging task. The primary difficulty is that the measure $r(Q)$ is *non-additive*. In other words, BCs of two distinct nodes in Q cannot, in general, be simply added when computing $r(Q)$. This is obvious, since a specific path between nodes i and j, $i, j \in V \backslash Q$, with two or more internal nodes in Q is counted only once in the computation of $r(Q)$.

We first show that PP and ESPP are strongly NP-hard. An efficient, polynomial-time algorithm for obtaining an optimal solution on general graphs is, therefore, unlikely. Then, we address special graphs (bicliques and balanced and full d-trees). Finally, we analyze a heuristic for general trees.

Proof of Hardness of PP and ESPP

Puzis et al. (2007) use the Vertex Cover Problem (Garey and Johnson 1979) to show the hardness of the non-normalized measure $BC(Q)$ (Section 2.2.2), which is the numerator of our measure $r(Q)$. Furthermore, given G and k, the variant considered in Puzis et al. (2007) requires that the solution has exactly k nodes. The strongly NP-Complete problem which we use in our reduction is the Independent Set Problem (Garey and Johnson 1979).

Independent Set Problem (ISP)

Instance. A connected, undirected graph $G = (V, E)$; a positive integer $k \leq |V|$.

Solution. A set of nodes, $I \subseteq V, |I| \geq k$, such that no two nodes in I are connected by an edge in E.

Theorem 2.3.5 *The decision problem corresponding to PP is strongly NP-Complete.*

Proof: Given an arbitrary instance of ISP, specified by $G(V, E)$, we consider the following decision problem:

DECISION QUESTION: Does there exist a portal $Q \subseteq V$ in $G(V, E)$ such that $|Q| \leq |V| - k$ and $r(Q) \geq 1$?

Note that the decision problem is clearly in class NP. We now show that ISP has an affirmative answer if and only if the above decision question has an affirmative answer.

Suppose I^* is an independent set in G with at least k^* nodes. Let $Q^* = V \setminus I^*$. Then, $|Q^*| \leq |V| - k^*$. From the definition of an independent set, it follows that all paths in G between any two nodes in I^* have at least one node in Q^* as an internal node. Thus, $r(Q^*) = 1$ and the decision question has an affirmative answer. Conversely, if there exists $Q \subseteq V$ with $|Q| \leq |V| - k$ and $r(Q) \geq 1$, then the set $V \setminus Q$ is an independent set of at least k nodes. ∎

Corollary 2.3.6 *The decision problem corresponding to ESPP is strongly NP-Complete.*

Results on Specific Families of Graphs

We now discuss two specific families of graphs: Bicliques and Balanced and Full d-Trees.

Bicliques:

Let $G = (U \cup V, E)$ be a biclique: $n_1 = |U| \leq |V| = n_2$ and $u \in U, v \in V$ implies that $\{u, v\} \in E$. The *size* of the biclique is $n_1 + n_2$. Let $Q_1, Q_2 \subseteq U \cup V$. If $|Q_1 \cap U| = |Q_2 \cap U|$ and $|Q_1 \cap V| = |Q_2 \cap V|$, then $BC(Q_1) = BC(Q_2)$. Thus, for $Q \subseteq U \cup V$, the objective function $r(Q)$ depends only on two numbers: $k_1 = |Q \cap U|$ and $k_2 = |Q \cap V|$. Theorem 2.3.7 (resp., Corollary 2.3.11) provides an optimal solution to PP (resp., ESPP).

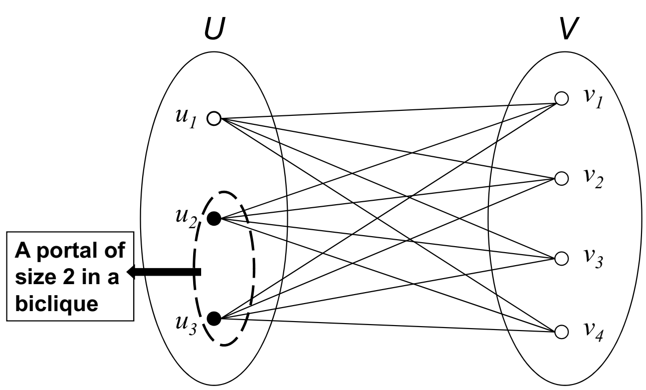

Figure 2.8. Optimal Portal in a Biclique.

Theorem 2.3.7 *Let $G = (U \cup V, E)$ be a biclique with $n_1 = |U| \leq |V| = n_2$. Let $Q \subseteq U \cup V$. Let $k_1 = |Q \cap U|$, $k_2 = |Q \cap V|$. Then,*

(i) For $1 \leq k \leq n_1 - 1$, any set Q which satisfies $k_1 = k$ and $k_2 = 0$ is an optimal solution of PP.

(ii) For $k \geq n_1$, then $Q = U$ is an optimal solution of PP.

Proof: The result of Theorem 2.3.7 follows from the proofs of Lemmas 2.3.8, 2.3.9, and 2.3.10.

Lemma 2.3.8 *Let $G = (U \cup V, E)$ be a biclique with $n_1 = |U| \leq |V| = n_2$. If $k \geq n_1$, then $Q = U$ is an optimal solution of PP.*

Proof: Let $Q = U$. Since $k \geq n_1$, we have $|Q| = |U| = n_1 \leq k$. Let $G'(Q)$ denote the induced subgraph obtained by removing all the nodes in Q from G. Thus, we have $G'(Q) = V$. For any two nodes in V, there exist n_1 shortest paths (in G) between them. Each shortest path has exactly one node in U as an internal node. Thus,

$$BC(Q) = \begin{pmatrix} |V| \\ 2 \end{pmatrix} \frac{n_1}{n_1} = \begin{pmatrix} n_2 \\ 2 \end{pmatrix}.$$

Also,

$$\begin{pmatrix} n_1 + n_2 - |Q| \\ 2 \end{pmatrix} = \begin{pmatrix} n_2 \\ 2 \end{pmatrix}.$$

Therefore, $r(Q) = 1$ and $Q = U$ is an optimal solution of PP. ■

To obtain an optimal solution of PP for $1 \leq k \leq n_1 - 1$, we first obtain an optimal solution of the corresponding instance of ESPP in the following lemma.

Lemma 2.3.9 *Let* $G = (U \cup V, E)$ *be a biclique with* $n_1 = |U| \leq |V| = n_2$*. For* $1 \leq k \leq n_1 - 1$*, any set* Q *with* $k_1 = |Q \cap U| = k$ *and* $k_2 = |Q \cap V| = 0$ *is an optimal solution of ESPP.*

Proof: Since n_1, n_2 and k are given, the value $\begin{pmatrix} n_1 + n_2 - k \\ 2 \end{pmatrix}$ is fixed. Thus, maximizing $r(Q)$ is equivalent to maximizing $BC(Q)$. We have

$$BC(Q) = \begin{pmatrix} n_1 - k_1 \\ 2 \end{pmatrix} \frac{k_2}{n_2} + \begin{pmatrix} n_2 - k_2 \\ 2 \end{pmatrix} \frac{k_1}{n_1}$$

Using $k_2 = k - k_1$, we obtain

$$\begin{aligned} BC(Q) &= \begin{pmatrix} n_1 - k_1 \\ 2 \end{pmatrix} \frac{k - k_1}{n_2} + \begin{pmatrix} n_2 - (k - k_1) \\ 2 \end{pmatrix} \frac{k_1}{n_1} \\ &= \frac{n_1(n_1 - k_1)(n_1 - k_1 - 1)(k - k_1) + n_2(n_2 - k + k_1)(n_2 - k + k_1 - 1)k_1}{2n_1 n_2} \end{aligned}$$

Let $g(k_1)$ represent the numerator of $BC(Q)$. Thus, $g(k_1) = n_1(n_1 - k_1)(n_1 - k_1 - 1)(k - k_1) + n_2(n_2 - k + k_1)(n_2 - k + k_1 - 1)k_1$. In the following claim, we show that for $0 \leq k_1 \leq k$, $g(k_1)$ reaches its maximum at $k_1 = k$.

Claim 2.3.6 *Let* $n_1 \leq n_2$ *and* $1 \leq k \leq n_1 - 1$*. For* $0 \leq k_1 \leq k$*,* $g(k_1) = n_1(n_1 - k_1)(n_1 - k_1 - 1)(k - k_1) + n_2(n_2 - k + k_1)(n_2 - k + k_1 - 1)k_1$ *reaches its maximum at* $k_1 = k$*.*

Proof: By defining $a = n_2 - n_1$, $b = n_1(2n_1 + k - 1) + n_2(2n_2 - 2k - 1)$, $c = n_1(-n_1^2 - 2kn_1 + n_1 + k) + n_2(n_2^2 - 2kn_2 - n_2 + k^2 + k)$, and $e = kn_1^2(n_1 - 1)$, we can rewrite $g(k_1)$ as $ak_1^3 + bk_1^2 + ck_1 + e$. Although k_1 can only take integral values, it is convenient to assume $g(k_1)$ as a continuous function of k_1 ($0 \leq k_1 \leq k$) for the purpose of this proof. We have two forms of $g(k_1)$ based on the value of a.

(i) For $n_1 < n_2$, we have $a > 0$. Thus, $g(k_1)$ is a cubic function of k_1. Let $g'(k_1)$ represent the first-order derivative of $g(k_1)$ with respect to k_1. Thus, $g'(k_1) = \frac{dg(k_1)}{dk_1} = 3ak_1^2 + 2bk_1 + c$ has at most two real roots, which correspond to the critical points of $g(k_1)$. Since $a > 0$, $g'(k_1)$ is convex. Let $\Delta = (2b)^2 - 4(3a)c$ represent the discriminant of $g'(k_1)$.

(1) For $\Delta \leq 0$, we have $g'(k_1) \geq 0$. Thus, $g(k_1)$ monotonically increases with k_1 and reaches its maximum at $k_1 = k$ for $0 \leq k_1 \leq k$.

(2) For $\Delta > 0$, $g'(k_1)$ has two real roots: $\bar{k}_1$ and $\hat{k}_1$. Without loss of generality, we assume $\bar{k}_1 < \hat{k}_1$. Thus, $\bar{k}_1 = \frac{-2b-\sqrt{\Delta}}{6a}$, $\hat{k}_1 = \frac{-2b+\sqrt{\Delta}}{6a}$. Let $g''(k_1) = \frac{d^2g(k_1)}{dk_1^2} = 6ak_1 + 2b$. Then, we have $g''(\bar{k}_1) = -\sqrt{\Delta} < 0$, $g''(\hat{k}_1) = \sqrt{\Delta} > 0$. Thus, $g(k_1)$ achieves its local maximum at $\bar{k}_1$ and achieves its local minimum at $\hat{k}_1$. Therefore, the maximum of $g(k_1)$ for $0 \leq k_1 \leq k$ has three possible values: $g(0)$, $g(k)$, or $g(\bar{k}_1)$ (if $0 \leq \bar{k}_1 \leq k$).

Since $k \leq n_1 - 1 < n_2 - 1$, we have $2n_2 - 2k - 1 > 0$. Since $n_1 > 0, n_2 > 0$, and $2n_1 + k - 1 > 0$, we have $b = n_1(2n_1 + k - 1) + n_2(2n_2 - 2k - 1) > 0$. Since $a > 0$ and $b > 0$, we have $\bar{k}_1 = \frac{-2b-\sqrt{\Delta}}{6a} < 0$. Thus, since $\bar{k}_1$ is outside the range $[0, k]$, the maximum of $g(k_1)$ for $0 \leq k_1 \leq k$ is either $g(0)$ or $g(k)$. Also, we have $g(0) = n_1^2(n_1 - 1)k$ and $g(k) = n_2^2(n_2 - 1)k$. Since $n_1 < n_2$, we have $g(0) < g(k)$. Therefore, for $0 \leq k_1 \leq k$, $g(k_1)$ reaches its maximum at $k_1 = k$.

(ii) If $n_1 = n_2$, we have $a = 0$, $b = n_1(4n_1 - k - 2)$, $c = n_1(-4kn_1 + k^2 + 2k)$, and $e = n_1^2(n_1 - 1)k$. Since $k \leq n_1 - 1$ and $n_1 \geq 1$, we have $4n_1 - k - 2 > 0$. Thus,

$b = n_1(4n_1 - k - 2) > 0$. Therefore, $g(k_1) = bk_1^2 + ck_1 + e$ is a quadratic function of k_1. Since $b > 0$, $g(k_1)$ is convex. Thus, the maximum of $g(k_1)$ for $0 \le k_1 \le k$ is either $g(0)$ or $g(k)$. Also, we have $g(0) = n_1^2(n_1 - 1)k = g(k)$. Therefore, for $0 \le k_1 \le k$, $g(k_1)$ reaches its maximum at $k_1 = k$.

This completes the proof of Claim 2.3.6. □

Thus, given an instance of ESPP with n_1, n_2, and k, the function $g(k_1)$ reaches its maximum at $k_1 = k$. Consequently, $r(Q)$ is maximized for any set Q with $k_1 = k, k_2 = 0$. This completes the proof of Lemma 2.3.9. ■

Lemma 2.3.9 provides an optimal solution of ESPP for $1 \le k \le n_1 - 1$. In our next result, we show that the optimal value $r(Q)$ of ESPP increases with $|Q|$ for $1 \le |Q| \le n_1 - 1$. Thus, given k, $1 \le k \le n_1 - 1$, an optimal solution of PP can be obtained by solving the corresponding instance of ESPP.

Lemma 2.3.10 *Let $G = (U \cup V, E)$ be a biclique with $n_1 = |U| \le |V| = n_2$. Let Q^* denote an optimal solution of an instance of PP defined by G and a positive integer k, $1 \le k \le n_1 - 1$. Then, $|Q^*| = k$. Consequently, an optimal solution of PP can be obtained by solving the instance of ESPP corresponding to G and k.*

Proof: Let $\hat{Q}$ be an optimal solution of an instance of ESPP defined by G and a positive integer $\hat{k}$. Thus, we have $|\hat{Q}| = \hat{k}$. We next show $r(\hat{Q})$ increases with $\hat{k}$. From Lemma 2.3.9, we have $\hat{k}_1 = |\hat{Q} \cap U| = \hat{k}$ and $\hat{k}_2 = |\hat{Q} \cap V| = 0$. Thus,

$$BC(\hat{Q}) \quad = \quad \begin{pmatrix} n_1 - \hat{k} \\ 2 \end{pmatrix} \frac{0}{n_2} + \begin{pmatrix} n_2 \\ 2 \end{pmatrix} \frac{\hat{k}}{n_1} = \frac{n_2(n_2 - 1)\hat{k}}{2n_1},$$

$$r(\hat{Q}) = \frac{BC(\hat{Q})}{\binom{n_1+n_2-\hat{k}}{2}}.$$

For $1 \leq \hat{k} \leq n_1 - 1$, the value $BC(\hat{Q})$ increases with $\hat{k}$, and $\binom{n_1+n_2-\hat{k}}{2}$ decreases with $\hat{k}$. Thus, $r(\hat{Q})$ increases with $\hat{k}$. Since Q^* is an optimal solution of an instance of PP defined by G and k, we have $r(Q^*) = \max_{1 \leq \hat{k} \leq k} r(\hat{Q})$. Since $r(\hat{Q})$ increases with $\hat{k}$, we have $|Q^*| = k$, $|Q^* \cap U| = k$, and $|Q^* \cap V| = 0$. The result follows. ∎

Together, Lemmas 2.3.8, 2.3.9, and 2.3.10, complete the proof of Theorem 2.3.7. ∎

We also summarize the solution of ESPP.

Corollary 2.3.11 *Let $G = (U \cup V, E)$ be a biclique with $n_1 = |U| \leq |V| = n_2$. Let $Q \subseteq U \cup V$. Let $k_1 = |Q \cap U|$, $k_2 = |Q \cap V|$.*

1. *For $1 \leq k \leq n_1 - 1$, any set Q which satisfies $k_1 = k$ and $k_2 = 0$ is also an optimal solution of ESPP.*

2. *For $n_1 \leq k \leq n_1 + n_2 - 1$, then any set Q which satisfies $k_1 = n_1$ and $k_2 = k - n_1$ is an optimal solution of ESPP.*

Balanced and Full d-Trees:

Given a tree $G(V, E)$ and $Q \subseteq V$, let $G'(Q)$ denote the induced subgraph obtained by removing all the nodes in Q from G. In general, $G'(Q)$ is a forest with disjoint trees as its connected components. Since G is a tree, there is a unique path in G connecting any two distinct nodes s and t in $V \backslash Q$; thus, $\sigma_{st} = 1$ (see Section 2.2.2). We first define some notation for a general tree $G(V, E)$:

n: the number of nodes in G (i.e., $n = |V|$).

k: the number of nodes in Q (i.e., $k = |Q|$).

l: the number of connected components in $G'(Q)$.

A_i: the i^{th} connected component in $G'(Q)$, $i = 1, 2, \cdots, l$.

a_i: the *size* (i.e., the number of nodes) of component A_i, $i = 1, 2, \cdots, l$.

Consider a connected component, say A_i, of $G'(Q)$. In G, there is a unique path from any node in A_i to each node in every other connected component in $G'(Q)$. Thus,

$$BC(Q) = \sum_{s \notin Q, t \notin Q, s \neq t} \frac{\sigma_{st}(Q)}{\sigma_{st}} = \sum_{1 \leq i < j \leq l} a_i a_j \tag{2.1}$$

Since $\sum_{i=1}^{l} a_i = |V| - |Q| = n - k$, we have

$$BC(Q) = \frac{(n-k)^2 - \sum_{i=1}^{l} a_i^2}{2} \tag{2.2}$$

Thus, for fixed n and k, maximizing $BC(Q)$ is equivalent to minimizing $\sum_{i=1}^{l} a_i^2$. We next illustrate the solution of this problem for balanced and full d-trees.

On a rooted balanced and full d-tree, each node (except the leaf nodes) has d distinct successors, each node (except root) has a unique predecessor. All leaf nodes have the same distance (height) to the root node. For a d-tree, if we remove any node other than the root node and leaf nodes, we will add d more connected components into the remaining graph. So if we remove k nodes from a d-tree, we will have at most $l = dk + 1$ connected components left.

Theorem 2.3.12 *Let G be a balanced and full d-tree with height $h \geq 2$. For an instance of PP defined by G and a positive integer k, let $t = \min\{\lceil h/2 \rceil, \lfloor \log_d k \rfloor\}$ and let $\bar{Q}$ denote the*

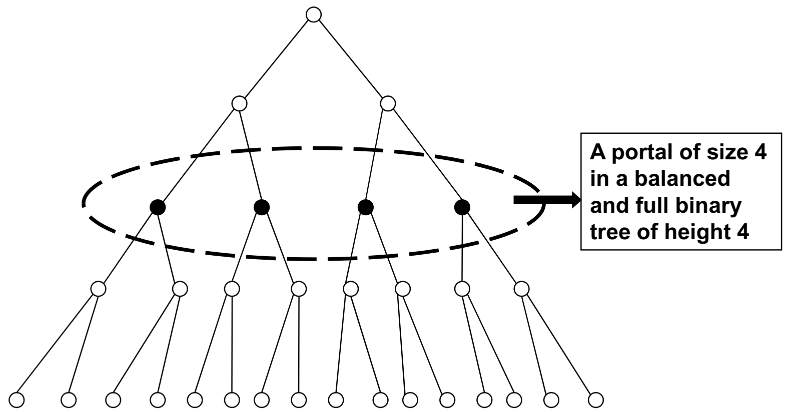

Figure 2.9. Optimal Portal in a Balanced and Full Binary (d=2) Tree.

set of nodes on the t^{th} level of G. Then, $\bar{Q}$ provides an asymptotically maximal solution to PP, with $r(\bar{Q}) \geq (1 - \frac{1}{d^{t+1}})$.

Proof: First, since the t^{th} level of a balanced and full d-tree has d^t nodes, $|\bar{Q}| = d^t \leq k$. Note that $G'(\bar{Q})$ has exactly $l = d|\bar{Q}| + 1$ connected components. Of these, we have (i) $d|\bar{Q}|$ identical components, say $A_i, i = 1, 2, \ldots, d|\bar{Q}|$, each with $\frac{d^{h-t}-1}{d-1}$ nodes. Thus, $a_1 = a_2 = \cdots = a_{d|\bar{Q}|} = \frac{d^{h-t}-1}{d-1}$, and (ii) one component, say A_l, with $\frac{d^t-1}{d-1}$ nodes. Thus, $a_l = \frac{d^t-1}{d-1}$.

From Equation 2.1, we have

$$\begin{aligned} BC(\bar{Q}) &= \sum_{1 \leq i < j \leq l} a_i a_j \\ &= d|\bar{Q}|a_l a_1 + a_1^2 \begin{pmatrix} d|\bar{Q}| \\ 2 \end{pmatrix} \\ &= d^{t+1}\frac{d^t-1}{d-1}\frac{d^{h-t}-1}{d-1} + \frac{(d^{h-t}-1)^2}{(d-1)^2}\frac{d^{t+1}(d^{t+1}-1)}{2} \\ &= \frac{d^{t+1}(d^{h-t}-1)(d^{h+1}-d^{h-t}-d^{t+1}+2d^t-1)}{2(d-1)^2}. \end{aligned}$$

Also, $\begin{pmatrix} n - |\bar{Q}| \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{d^{h+1}-1}{d-1} - d^t \\ 2 \end{pmatrix} = \frac{(d^{h+1}-d^{t+1}+d^t-1)(d^{h+1}-d^{t+1}+d^t-d)}{2(d-1)^2}$.

Thus,

$$
\begin{aligned}
& BC(\bar{Q}) - (1 - \frac{1}{d^{t+1}}) \begin{pmatrix} n - |\bar{Q}| \\ 2 \end{pmatrix} \\
= \quad & \frac{1}{2d^{t+1}(d-1)^2} [d^h(d^{t+3} - d^{t+2} + 2d^{t+1} - d^2 - d) \\
& + d - d^t + d^{2t} - d^{2t+1} + 2d^{2t+2} - d^{2t+3} - d^{3t+1}]
\end{aligned}
$$

Since $t = \min\{\lceil h/2 \rceil, \lfloor \log_d k \rfloor\}$ and $\lceil h/2 \rceil \leq (h+1)/2$, we have $t \leq \lceil h/2 \rceil \leq (h+1)/2$. Thus, $d^h \geq d^{2t-1}$. Since $(d^{t+3} - d^{t+2} + 2d^{t+1} - d^2 - d) \geq 0$, we have

$$
\begin{aligned}
BC(\bar{Q}) - (1 - \frac{1}{d^{t+1}}) \begin{pmatrix} n - |\bar{Q}| \\ 2 \end{pmatrix} \quad \geq \quad & \frac{1}{2d^{t+1}(d-1)^2} [d^{2t-1}(d^{t+3} - d^{t+2} + 2d^{t+1} - d^2 - d) \\
& + d - d^t + d^{2t} - d^{2t+1} + 2d^{2t+2} - d^{2t+3} - d^{3t+1}] \\
= \quad & \frac{(d^t - d)(d^{2t}(d(d-2)+2) - 1)}{2d^{t+1}(d-1)^2} \\
\geq \quad & 0
\end{aligned}
$$

Thus,

$$
r(\bar{Q}) = \frac{BC(\bar{Q})}{\begin{pmatrix} n - |\bar{Q}| \\ 2 \end{pmatrix}} \geq (1 - \frac{1}{d^{t+1}}).
$$

■

Since $t = \min\{\lceil h/2 \rceil, \lfloor \log_d k \rfloor\}$, the ratio $r(\bar{Q}) \to 1$ with an increase in the size of G and k. Thus, the resulting family of solutions $\bar{Q}$ can be referred to as an asymptotically maximal family. Note that the solution $\bar{Q}$ is not necessarily optimal. Thus, in cases where $\bar{Q}$ is not optimal, there can exist a solution that is superior to $\bar{Q}$.

Remark 2: The solutions of PP and ESPP on paths, cycles, and cliques, are straightforward to obtain. Therefore, these results are listed without proofs as follows.

Theorem 2.3.13 *Let G be a path v_1-v_2-$\ldots$-v_n and consider an instance of ESPP specified by G and k. Let $\mu = \frac{n-k}{k+1}, c = n-k-(k+1)\lfloor\mu\rfloor$. Then, Q^* is an optimal solution of ESPP if and only if $G'(Q^*)$ has exactly c connected components of size $\lfloor\mu\rfloor+1$ and exactly $(k+1-c)$ connected components of size $\lfloor\mu\rfloor$.*

Thus, given an explicit description of the path v_1-v_2-$\ldots$-v_n and a positive integer k, an optimal solution of ESPP is $Q^* = \{v_{i(\lfloor\mu\rfloor+2)}, i = 1, 2, \cdots, c\} \cup \{v_{c(\lfloor\mu\rfloor+2)+j(\lfloor\mu\rfloor+1)}, j = 1, 2, \cdots, k-c\}$. The optimal objective function value is $r(Q^*) = BC(Q^*)/\begin{pmatrix} n-k \\ 2 \end{pmatrix}$, where

$$BC(Q^*) = \frac{(n-k)^2 - c(\lfloor\mu\rfloor+1)^2 - (k+1-c)(\lfloor\mu\rfloor)^2}{2}.$$

An optimal solution of PP is also easy to obtain: we simply solve ESPP for each $\bar{k} \leq k$. Since $k \leq n$, this requires time polynomial in the size of the input (assuming that the path is explicitly specified in the input).

Theorem 2.3.14 *Let G be a cycle v_1-v_2-$\ldots$-v_n-v_1 and consider an instance of ESPP specified by G and k. Let $\mu = \frac{n-k}{k}, c = n-k-k\lfloor\mu\rfloor$. Then, Q^* is an optimal solution of ESPP if and only if $G'(Q^*)$ has exactly c connected components of size $\lfloor\mu\rfloor+1$ and exactly $(k-c)$ connected components of size $\lfloor\mu\rfloor$.*

Thus, given an explicit description of the cycle v_1-v_2-$\ldots$-v_n-v_1 and a positive integer $k \geq 2$, an optimal solution of ESPP is $Q^* = \{v_{i(\lfloor\mu\rfloor+2)}, i = 1, 2, \cdots, c\} \cup \{v_{c(\lfloor\mu\rfloor+2)+j(\lfloor\mu\rfloor+1)}, j =$

$1, 2, \cdots, k-c\}$. The optimal objective function value is $r(Q^*) = BC(Q^*)/\begin{pmatrix} n-k \\ 2 \end{pmatrix}$, where

$$BC(Q^*) = \frac{(n-k)^2 - c(\lfloor \mu \rfloor + 1)^2 - (k-c)(\lfloor \mu \rfloor)^2}{2}.$$

We can easily obtain an optimal solution of PP by solving ESPP for each $\bar{k} \leq k$. Since $k \leq n$, this requires time polynomial in the size of the input (assuming that the cycle is explicitly specified in the input).

Let $G = (V, E)$ be a clique and let $Q \subseteq V$. For any two nodes $s, t \in V \backslash Q$, the unique shortest path in G between s and t is of length 1 and exists in $G'(Q)$. Thus, no shortest path between any two nodes in $V \backslash Q$ has an internal node in Q. Thus, $\sigma_{st}(Q) = 0$. It follows that $r(Q) = BC(Q) \equiv 0$ for any $Q \subseteq V$. In other words, any subset of nodes is an optimal solution for PP and ESPP.

General Trees

An important question that arises naturally is that of the solution of PP on a general tree. The tree structure occurs frequently in real-world social networks. Wein (2009) describes the milk supply chain as a tree. The author argues that for a potential terrorist attack, it is enough to introduce a small amount of toxin (botulinum) at a few key nodes of the tree. We believe that such nodes can be identified by finding an optimal or near-optimal portal in the tree. In Perer and Wilson (2007), the authors discuss the underground distribution network of steroids among players of Major League Baseball. Investigators have used this network to determine the role of each individual in the distribution of steroids. Again, knowledge of a good portal in this network should help identify key members. Interfirm collaboration networks, studied in Schilling and Phelps (2007), are approximately trees. In their search for firms with a higher innovative output, the authors find a set of nodes that is a near-optimal portal. In Hanaki et al. (2007), the authors argue that locally, a large and sparse random network often resembles a pure branching tree.

The computational complexity of PP for a tree is an open problem. We now present a simple heuristic (Theorem 2.3.15) to find a portal in a tree and obtain a lower bound on its performance. To describe the heuristic, we need the following labeling procedure:

Labeling Procedure

Input: A tree $G(V, E)$.

Initialization: Let $i = 0$.

Step 1: Select all the leaf nodes of G, label them as being on Level i, and include them in set $S(i)$. Let $G = G \backslash S(i)$.

Step 2: If $G = \emptyset$, terminate; otherwise, let $i = i + 1$ and go to Step 1.

We record the highest level we get in this labeling procedure as h, and refer to it as the *height* of the tree. Let $n[j] = |S(j)|$, the number of nodes in Level j. It is easy to see that $n[0] \geq n[1] \geq n[2] \geq \ldots \geq n[h]$ and $n[h] \in \{1, 2\}$.

Theorem 2.3.15 *Let G be a tree with height $h \geq 2$. For an instance of PP defined by $G(V, E)$ and a positive integer $k \geq 2$,*

1. *If $n[1] \leq k$, then set $\bar{t} = 1$.*

2. If $n[1] > k$, then set $\bar{t}$ such that $n[\bar{t}-1] > k, n[\bar{t}] \leq k$.

Let $t = \max\{\lceil h/2 \rceil, \bar{t}\}$. Let $\bar{Q}$ denote the set of nodes on the level t of G. Then, $\bar{Q}$ provides a solution to PP with $r(\bar{Q}) \geq \frac{b(b-1)t^2+2bt(h-t)}{(n-b)(n-b-1)}$, where $b = n[t]$ and $n = |V|$. Moreover, this bound is tight.

Proof: Since $\bar{Q}$ includes all the nodes on level t of G, we have $|\bar{Q}| = n(t)$. Since $t = \max\{\lceil h/2 \rceil, \bar{t}\} \geq \bar{t}$, we have $n(t) \leq n[\bar{t}] \leq k$. Thus, $|\bar{Q}| \leq k$. Note that $G'(\bar{Q})$ has $l \geq b+1$ connected components. Of these, we have (i) at least b components, say $A_i, i = 1, 2, \ldots, b$, each with at least t nodes: from level 0 to level $t-1$, A_i has at least one node from each level. Thus, $a_i = |A_i| \geq t, i = 1, 2, \ldots, b$, and (ii) one component, say A_l, with at least $h-t$ nodes: from level $t+1$ to level h, A_l has at least one node from each level. Thus, $a_l = |A_l| \geq h - t$. From (2.1) defined above, we have

$$\begin{aligned} BC(\bar{Q}) &= \sum_{1 \leq i < j \leq l} a_i a_j \\ &\geq \sum_{1 \leq i < j \leq b} a_i a_j + \sum_{1 \leq i \leq b} a_i a_l \\ &\geq \binom{b}{2} t^2 + bt(h-t) \\ &= \frac{b(b-1)}{2} t^2 + bt(h-t). \end{aligned}$$

Also, $\binom{n-|\bar{Q}|}{2} = \frac{(n-b)(n-b-1)}{2}$. Thus, we have

$$r(\bar{Q}) = \frac{BC(\bar{Q})}{\binom{n-|\bar{Q}|}{2}} \geq \frac{b(b-1)t^2 + 2bt(h-t)}{(n-b)(n-b-1)}.$$

To show the tightness of the bound, let G be a path of length 4. Thus, $n = 5$, $h = 2$, $n[0] = n[1] = 2$, $n[2] = 1$. For $k = 2$, if we apply the heuristic, we have $\bar{Q}$ as all the nodes on Level 1 of G. Then, $t = 1, b = n[1] = 2$, $\frac{b(b-1)t^2+2bt(h-t)}{(n-b)(n-b-1)} = 1$, which implies that $\bar{Q}$ is an optimal solution to PP on G. ■

When G is a balanced and full d-tree, the procedure in Theorem 2.3.15 is the same as the one in Theorem 2.3.12, which was shown to provide an asymptotically maximal solution to PP.

2.4 Conclusions and Future Research Directions

The ability to find useful structures in social networks will undoubtedly benefit their users as well as other stakeholders – the businesses that use these networks and the sites that host them. Unlike the internet, structural search on social networks is set-based and offers a rich variety of interesting combinatorial optimization problems. In this paper, our effort is to identify and analyze specific instances of such problems. We consider two problems – the Elite Group Problem (EGP) and the Portal Problem (PP) – derived, respectively, from the notions of influence and centrality. We demonstrate the relevance of these problems on a variety of social networks and show the following results: (i) the basic EGP is polynomially solvable while its size-constrained variant is strongly NP-hard. We also show the hardness of a penalty-based relaxation of the size-constrained version. (ii) PP is strongly NP-hard. We discuss the solution of PP on several special networks – bicliques, balanced and full d-trees, paths, cycles, and cliques – and propose a heuristic for general trees.

In the industry, the focus, thus far, has been on developing "social search engines" to search social media and user-generated content, e.g., Social Mention (http://www.social mention.com), Twitter (http://search.twitter.com), and Delver (http://www.delver.com). Some networks do facilitate simple search, e.g., MySpace allows a user to find other users with similar interests. However, to our knowledge, there is little or no sophisticated structural search available to ordinary users of social networks. Since this type of search is typically combinatorial in nature, the resulting problems are expected to be challenging. One idea is to provide an easy-to-use modeling language to enable members to specify complex, constrained search and then use sophisticated solvers (e.g., CPLEX) or heuristics to solve the resulting problems. Another possibility is to develop a repository – that could evolve over time – of efficient algorithms for the typical combinatorial searches that users specify. The notions of an elite group and a portal studied in this paper are extensions to set-based measures of, respectively, indegree and betweenness centralities for individual members of a social network. Similarly, useful structures based on extensions of other popular centralities, e.g., the more general degree centrality or closeness centrality (Carrington et al. 2005), could also be investigated. Applications of such set-based measures have been discussed for several social networks (see, e.g., Cattani and Ferriani 2008, Owen-Smith et al. 2002, Morselli and Giguere 2006).

The ideas of search developed in this paper naturally flow into other operational problems of interest. One such problem is targeted online advertising. For example, in the Twitter network, it is possible for one member to "follow" another, suggesting a directed link in the network. The identification of an elite group within Twitter could, therefore, be used to target promotional material to members of this group. For instance, an advertisement could be targeted using keywords exchanged by two members during a conversation on Twitter.

CHAPTER 3

VALUE OF LOCAL CASH REUSE: INVENTORY MODELS FOR MEDIUM-SIZE DEPOSITORY INSTITUTIONS UNDER THE NEW FEDERAL RESERVE POLICY [1]

3.1 Synopsis

Unlike most consumer products, a significant portion of physical cash is recirculated and reused in an economy. The Federal Reserve System (Fed) of the United States provides currency services (e.g., separating cash into notes that are fit and unfit for recirculation) to depository institutions (DIs; e.g., banks) who, in turn, meet the public's demand for cash. Over the years, the Fed's analysis has suggested that DIs are relying on the Fed's cash processing services to recirculate a substantial amount of physical cash that the DIs should have recirculated themselves (Federal Reserve 2003, 2006). As identified by the Fed, it is often the case that cash deposited to the Fed by a DI is reordered by the same DI in the same denomination within one business week. From the point of view of a DI, this practice – referred to as *cross-shipping* – made sense (prior to 2007) since the Fed's cash processing services were free. Therefore, it was convenient and economical for a DI to let the Fed clean the cash, i.e., separate it into fit and unfit notes. This practice, however, progressively and significantly increased the Fed's expenses to process cash and manage its currency circulation

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operations (Blacketer and Evetts 2004). With the aim of encouraging DIs to "locally" clean and reuse cash to ultimately reduce the societal cost of providing currency to the public, the Fed introduced its new currency recirculation policy in July 2007. The details of the Fed's policy will be discussed in Section 3.1.1.

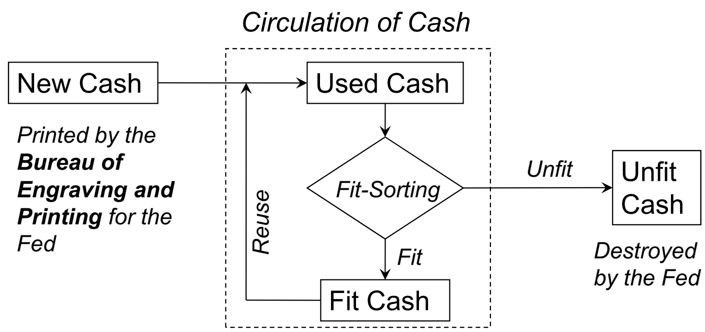


Figure 3.1. Life Cycle of Physical Cash.

In its physical form, cash can be classified into four categories (Figure 3.1). The Bureau of Engraving and Printing (BEP) prints *new cash* on the Fed's request. After its introduction into the economy, this cash is referred to as *used cash*. Over time, the quality of used cash deteriorates. Used cash that is deemed suitable to remain in circulation is referred to as *fit cash*. The remainder is *unfit cash*, which is typically worn, soiled, or torn, and is destroyed by the Fed. In this paper, we only address the management of physical notes. Coins are not considered since they are less perishable, have lesser value, and the guidelines and challenges in managing their use are markedly different from those of notes. We now describe the structure of the U.S. cash supply chain.

3.1.1 The Closed-Loop Cash Supply Chain

As in Rajamani et al. (2006), it is convenient to describe the cash supply chain as a closed-loop supply chain using its 4-P components: Product, Players, Process, and Policy.

- **Product:**

  The Fed's new currency recirculation policy applies only to the \$10 and \$20 notes (Federal Reserve 2006). Between these two denominations, \$20 notes form an overwhelmingly dominant portion of the currency in circulation because of their heavy use in ATMs. Consequently, DIs are primarily interested in recirculating \$20 notes (Federal Reserve 2006). Our analysis, therefore, assumes currency of a single denomination, namely \$20. Both the transportation and transactions of cash between DIs and the Fed occur in integer multiples of cash *bundles*, where a bundle is 1000 notes of the same denomination.

- **Players & Process:**

  The Fed acts as the exclusive wholesaler, that orders the production of cash from the BEP and controls the release of cash into the economy through DIs. The DIs act as retailers, providing cash to end customers and collecting deposits from them. Individuals and Commerce are the end customers, who deposit used cash to and/or withdraw fit cash from the DIs through various channels such as ATMs, bank branches, and businesses. Besides these three primary players, third-party logistic providers (3PLPs) such as Brink's, Inc., Dunbar Armored, and Loomis, Fargo & Co., also play an important role in this supply chain. In addition to collecting cash and transporting it between the DIs and the Fed, the 3PLPs are also involved in the processing of cash, e.g., counting and sorting cash into fit and unfit notes. Before July 2007, the Fed accepted deposits of used cash from DIs, fit-sorted that currency (i.e., separated it into fit and unfit cash), removed unfit cash from circulation, and provided fit cash to DIs.

The DIs simply sorted the collected used cash by denomination and then forwarded it to the Fed. After the implementation of the Fed's policy, in an ideal situation, the DIs locally fit-sort the used cash and use the resulting fit cash to satisfy (partially or fully) their customers' demand. From the systemic view, the local reuse of fit cash by DIs can be an influential practice because a significant fraction – about 65-75%, based on the Fed's documentation (Federal Reserve 2010b) and our conversations with a third-party provider – of used cash is fit cash. Thus, a major percentage of the movement of used cash (resp., fit cash) to (resp., from) the Fed can be avoided and, consequently, help in supporting the Fed's eventual goal of minimizing the cost of providing currency to the public. Our analysis in this paper assumes that the percentage of fit cash in the used cash is 75%.

The inventory models in this paper specifically capture the operations of DIs that handle a medium volume of currency. Such DIs face a weekly demand of fit cash that is typically between 1000 to 2000 bundles. The weekly demand at DIs that handle large volumes typically averages more than 5000 bundles (Dawande et al. 2010). Due to the relatively low volume of physical cash, medium DIs may prefer to engage the services of 3PLP for fit-sorting and managing custodial inventory (to be defined shortly). This has been anticipated by the Fed as well (Federal Reserve 2003). Furthermore, due to the relatively high percentage variation (as compared to large DIs) in the weekly transaction of cash, such DIs are expected to engage the services of the 3PLP on an as-required basis (as opposed to a daily or weekly service, which is the typical case with large DIs). In turn, since service requests from such a DI can be irregular, a

3PLP typically uses a fixed-fee structure: a fixed-cost for ordering transportation, a setup cost for fit-sorting in addition to a per-bundle fit-sorting cost, and a setup cost for withdrawing cash from custodial inventory. Since the deposit (if any) into custodial inventory occurs immediately after fit-sorting (for which a fee is charged), 3PLPs typically do not include a separate fee for such deposits.

- **Policy:**

As mentioned earlier, DIs had been overusing (prior to 2007) the cash processing services of the Fed via the practice of cross-shipping. There were two main reasons for DIs to cross-ship. First, the cost of fit-sorting – regardless of whether the DI invests in the infrastructure itself or uses a 3PLP – discourages DIs from locally reusing cash: it requires sophisticated labor and expensive machines. On the other hand, the Fed provided this service for free. Second, for a DI, currency held on account at the Fed can earn interest (by lending to other institutions via the Federal Funds Market), while currency held in its own vault cannot. Therefore, holding cash incurs a cost of lost opportunity for a DI. Hence, it was natural for a DI to minimize the volume of cash held in its own vault while still satisfying customers' demand. To achieve this most efficiently, DIs frequently deposited/withdrew currency to/from the Fed, which resulted in cross-shipping.

The concerns over cross-shipping, and the consequent increase in the amount spent on its currency management operations, motivated the Fed to introduce the new currency recirculation policy in July 2007. The policy has two primary components: a recirculation fee on cross-shipped currency and the custodial inventory program.

> **Recirculation Fee:** (Federal Reserve 2006) [Federal Reserve System] proposes to establish a recirculation fee to provide further incentive for depository institutions to recirculate currency. Depository institutions would pay the fee when they deposit [fit or non-fit-sorted cash] and order the same denomination within the same business week [within] a Reserve Bank zone or subzone. The fee would not be [activated by] deposits of... [unfit] cash.

Currently, the Fed charges a per-bundle cross-shipping penalty of $5. The volume of cross-shipping is calculated as the lesser of (1) the fit cash in the used-cash deposit and (2) the fit-cash withdrawal in the same week. Thus, the current recirculation fee is the volume (i.e., number of bundles) of cross-shipping multiplied by five.

> **Custodial Inventory Program:** (Federal Reserve 2006) A custodial inventory enables an institution to transfer currency to the Federal Reserve Bank's books, but physically hold the currency within their secured facility, thereby reducing the investment cost of holding currency long enough to recirculate it to customers. The Custodial Inventory program is applicable only to $10 and $20 notes.

The program encourages fit-sorting by DIs as the currency held in custodial inventory is eligible to be lent to other institutions (and is, hence, an earning asset) and is more readily available to meet customer demand than fit cash ordered from the Fed. Note that the program does not allow a DI to deposit cash withdrawn from the Fed into custodial inventory. In other words, only the cash that has been fit-sorted by the DI is eligible to be deposited into custodial inventory.[1]

Finally, it is important to note that a DI may choose to not fit-sort cash and, consequently, not participate in the custodial inventory program. Instead, the DI can choose to pay recirculation fee if it indulges in cross-shipping.

3.1.2 Literature Review and Our Contributions

There are three broad domains of research that are relevant to our work. We now briefly review them and outline our contributions along each direction.

- **Cash Supply Chain**: Since the Fed's recirculation policy has been recently implemented, there are only a few studies that analyze the operational issues resulting from the policy. Under the simplifying assumption that both the demand of fit cash and the deposit of used cash satisfy week-to-week periodicity, both Geismar et al. (2007) and Mehrotra et al. (2010) derive characteristics of optimal policies to manage the inventory of cash. In these papers, periodicity means that the demand (resp., deposit) of fit-cash (resp., used-cash) on Day i (say, Monday) is exactly the same *every* week. The periodicity assumption is typically reasonable for (large) DIs that transact a high volume of physical cash each week. Our models are more general in that they allow for arbitrary demands over a finite planning horizon. Furthermore, the analyses in both Geismar et al. (2007) and Mehrotra et al. (2010) are largely based on 2-week cyclic

[1]Apart from the description above, the Fed's policy includes some other operational details such as a *de minimis* exemption on cross-shipped cash. The *de minimis* exemption allows DIs to cross-ship a small volume of cash per quarter without any penalty. Moreover, the Fed intends to periodically review the *de minimis* exemption and may modify it at any time (Federal Reserve 2010c). Therefore, we have chosen not to include this exemption in our analysis. There are also some additional rules on operating custodial inventory. These details are secondary in nature and can be ignored for the purpose of assessing the overall impact of the Fed's policy.

policies. Such policies can be significantly suboptimal for medium-size DIs since such DIs typically need to schedule irregular and infrequent service requests due to relatively low volumes of cash transactions. Moreover, as mentioned earlier, our focus on medium-size DIs necessitates a significantly different cost structure for transportation and fit-sorting. We assume a fixed-cost for ordering transportation and a setup cost for fit-sorting in addition to a per-bundle fit-sorting cost, while both Geismar et al. (2007) and Mehrotra et al. (2010) assume simple per-bundle transportation and fit-sorting costs.

Rajamani et al. (2006) discuss the Fed's cash recirculation guidelines and describe the U.S. cash supply chain as a closed-loop supply chain. Dawande et al. (2010) examine the efficacy of the Fed's policy to coordinate the cash supply chain.

Our Contributions: As mentioned above, the assumptions used – and the solutions derived – in both Geismar et al. (2007) and Mehrotra et al. (2010) are not suitable for medium-size DIs. Therefore, our attempt in this paper has been to study inventory models that specifically capture the operations of medium-size DIs. We analyze two multi-period models motivated by a DI's aim to minimize the total cost incurred in managing the inventory of cash over a finite planning horizon. From a technical perspective, the non-periodicity and the different cost structure makes these models fundamentally different from those in Geismar et al. (2007) and Mehrotra et al. (2010). Therefore, a much more sophisticated analysis is required.

- **Remanufacturing and Reuse**: Guide (2000) surveys production planning and control activities at remanufacturing firms and discusses seven complicating characteristics with respect to traditional manufacturing firms. Among them, the issue of balancing product returns (in our case, used-cash deposits by the customers) with demand (in our case, fit-cash demand by the customers) is related to our analysis in this paper. Guide and van Wassenhove (2001) develop a framework for analyzing the profitability of reuse activities in remanufacturing. In our model, under the Fed's new policy, DIs self-select to implement the fit-sorting (i.e., reuse) activity if it is economically attractive for them. For false-failure returns, Ferguson et al. (2006) design contracts that incentivize a retailer to increase her effort and reduce the number of returns. We expand on the conceptual similarity between false-failure returns and reuse of cash in Section 3.7.

Our Contributions: Our work studies a novel closed-loop supply chain, namely the supply chain of physical cash. The structure and operations of the cash supply chain, e.g., the different classes of cash inventories (fit, used, and unfit cash) and the ability to convert one type of cash (used cash) into another (fit cash) via fit-sorting, make the setting quite unique. The Fed's recirculation policy is an attempt to improve the efficiency of the "reverse" channel in the cash supply chain.

- **Lot-Sizing Models**: The problems studied in this paper are related to the classical lot-sizing literature. Since the literature in this domain is quite extensive, we avoid providing a detailed review and, instead, only mention a few seminal studies: Wagner and Whitin (1958), DeMatteis and Mendoza (1968), Balintfy (1964), and Joneja (1990). Some excellent review papers in the lot-sizing domain are Robinson et al. (2009), Drexl and Kimms (1997), Kuik et al. (1994), and Salomon et al. (1991).

Our Contributions: We address new and challenging lot-sizing models that naturally arise in a real-world setting. The uniqueness of our models stems from the combined presence of the following aspects: (1) the different classes of cash inventories (used-, fit-, and unfit cash) and the partial convertibility of used cash into fit cash, via the process of fit-sorting, (2) the novel notion of cross-shipping, the non-linear definition of cross-shipping penalty, and the Fed's mechanism (i.e., a recirculation fee that is applicable only in the presence of cross-shipping), and (3) the fixed/variable costs charged by the secure-logistics provider together with the cross-shipping costs.

Given the Fed's policy, a DI has two options: it can either (i) continue to rely on the Fed for fit-sorting and incur cross-shipping penalties, or (ii) undertake fit-sorting and reuse (some or all of) its used-cash deposits. The Basic Model (BM), developed in Section 3.2, captures a DI's mode of operations if it chooses the first option, i.e., it implements neither fit-sorting nor the custodial inventory option. Our main result in this section is a polynomial-time dynamic programming algorithm for BM. Section 3.3 discusses the reuse model (RM), in which a DI chooses the second option and implements both the components of the Fed's policy. We prove that the decision problem corresponding to RM is NP-complete and formulate RM as a mixed-integer program. In Section 3.4, via a comprehensive numerical study, we evaluate the value of local reuse for a DI and develop insights into the impacts of the DI-specific parameters and the Fed's cross-shipping fee on the effective management of physical cash. Our primary conclusion is that the saving for a DI from implementing the two components of the Fed's policy as well as the extent of local recirculation are substantial. Section 3.5 develops a rolling-horizon procedure to adapt the decisions from our models for real-time solutions. In Section 3.6, we compare a DI's decisions under the Fed's policy with those under a socially-optimal mechanism. Finally, in Section 3.7, we summarize our results, discuss the limitations of our analysis, and outline future research directions.

3.2 The Basic Model (BM)

The Basic Model (BM) captures a DI's mode of operations if it implements neither fit-sorting nor the custodial inventory option and, instead, chooses to incur the cross-shipping penalty. We first provide a dynamic programming (DP) formulation of BM. Two important structural results (Theorems 3.2.1 and 3.2.3) allow us to significantly reduce the complexity of solving BM and show (in Theorem 3.2.5) that BM can be solved in time $O(T^3)$, where T is the length of the planning horizon. A generalized version of BM (referred to as BM^g), in which the initial and ending inventories of both fit-cash and used-cash can be arbitrary non-negative values, is discussed in Section 3.2.4.

3.2.1 Problem Formulation

As mentioned earlier, our objective is to minimize – over a given planning horizon – a DI's total cost of managing the inventory of cash, which includes (1) the ordering cost charged by the 3PLP, (2) the cross-shipping cost charged by the Fed and (3) the holding cost incurred by the DI. Ideally, an infinite horizon setting would be most appropriate, since DIs do operate under such an assumption. However, since (1) the periodicity assumption is not appropriate for medium-size DIs and (2) our interest is in precisely specifying the optimal weekly decisions under arbitrary (fit-cash) demands and (used-cash) deposits, we would need an explicit description of this data over an infinite horizon. On the other hand, considering a single-period (i.e., one week) model defeats any attempt to plan over a reasonable horizon. Therefore, as a tradeoff, we develop our solutions for an arbitrary finite horizon. In practice, DIs typically have good visibility for about one quarter into the future. Therefore, a choice of about 15 to 20 weeks would be appropriate for medium-size DIs to plan their decisions on managing physical cash inventory.

The planning horizon consists of T periods. The Fed charges its cross-shipping penalty if both withdrawal (of fit cash) and deposit (of used cash) occur within the same week. Therefore, it is natural for a DI to plan its transportation activities (corresponding to these withdrawals and deposits) on a per-week basis. For a medium-size DI, another auxiliary reason for choosing the length of a period as one week is that the total daily volume of transactions of physical cash is not high enough to warrant daily inventory decisions. As mentioned earlier, due to the low volume of cash, 3PLPs typically prefer a fixed-fee structure, which again supports the choice of per-week decisions on transportation, fit-sorting, and custodial inventory.

Next, note that a DI maintains precise records of transactions – both deposits and withdrawals – of physical cash for a considerable duration of time, for these are audited by the Fed. Thus, a DI has access to accurate and rich data on deposits and withdrawals. For medium-size DIs, while substantial week-to-week variations in the total weekly deposits and withdrawals are common, a relatively accurate (as compared to typical retail businesses) forecast can be projected, based on this data, over a modest planning horizon of, say, the next 10 to 15 weeks. Furthermore, as we will show later in Section 3.5, even in the presence of a considerable amount of variation between the forecasted and realized values of weekly deposits and withdrawals, the decisions based on forecasted data can be efficiently adapted to obtain near-optimal real-time solutions.

For the basic model, the **Timeline of the Activities** conducted by the DI is as follows.

- At the end of Period t, the inventories of used cash and fit cash are counted by the DI and decisions on (1) the quantity of used cash to be deposited to the Fed and (2) the quantity of fit cash to be withdrawn from the Fed are taken.
- The 3PLP transports the cash between the DI and the Fed. By the beginning of the Period $t+1$, the DI receives the fit cash withdrawn (if any) from the Fed.
- During Period $t+1$, the DI uses the fit cash inventory to satisfy demand (from customers) and collects the used cash deposited (by customers).

Notation

We now introduce the notation used for defining and analyzing the DP.

Parameters:

T	Number of periods in the planning horizon.
s	Transportation ordering cost, which is a fixed fee charged by the 3PLP.
h	Per bundle per period holding cost incurred by the DI.
e	Per bundle cross-shipping cost charged by the Fed.
ρ	The percentage of fit cash in the used cash. In our computations (Sections 3.4, 3.5, and 3.3.3), we set $\rho = 0.75$.
d_t^u	The projected deposit of used cash (in bundles) by the DI's customers during Period t, $t = 1, 2, \ldots, T$.
d_t^f	The projected demand of fit cash (in bundles) for the DI's customers during Period t, $t = 1, 2, \ldots, T$.

Decision Variables:

x_t^u	The quantity of used cash (in bundles) deposited to the Fed at the end of Period t, $t = 0, 1, \ldots, T$.
x_t^f	The quantity of fit cash (in bundles) withdrawn from the Fed at the end of Period t, $t = 0, 1, \ldots, T$.
X	the solution vector $(x_0^u, x_0^f, x_1^u, x_1^f, \ldots, x_T^u, x_T^f)$ of BM, which includes all the decisions made during the planning horizon.

Other Notation:

I_t^u	The inventory of used cash (in bundles) at the end of Period t, $t = 0, 1, \ldots, T$.
I_t^f	The inventory of fit cash (in bundles) at the end of Period t, $t = 0, 1, \ldots, T$.
v_t	$(I_t^u, I_t^f, x_t^u, x_t^f)$, the vector that denotes the state of the system and the decisions made at the end of Period t.
$G_t(I_t^u, I_t^f)$	The minimum cost incurred during periods 1 through t, given that, at the end of Period t, the inventory of used cash is I_t^u and the inventory of fit cash is I_t^f.
H_t	The holding cost incurred during Period t.
O_t	The ordering cost charged by the 3PLP for the transportation of cash at the end of Period $t-1$. Without loss of generality, this cost is charged to Period t.
R_t	The cross-shipping cost (if any) incurred on the transaction of used- and fit-cash between the DI and the Fed at the end of Period $t-1$. Without loss of generality, this cost is charged to Period t.
C_t	The total cost incurred during Period t. Thus, $C_t = H_t + O_t + R_t$.
$C(X)$	The total cost incurred during the entire T-period planning horizon for a solution X of BM.

A Dynamic Programming Formulation

We now define a dynamic programming formulation for BM. We assume that the initial and ending inventory for used cash and fit cash are both 0. Thus, we have the following boundary conditions:

$$x_0^u = I_0^u = I_0^f = 0, x_0^f > 0; x_T^u = I_T^u, x_T^f = I_T^f = 0.$$

Note that in reality, the initial and ending cash positions of a DI are never zero. The Fed requires DIs to hold a minimum of one day of average daily payments of fit cash in their vaults (Federal Reserve, 2006). Since this amount, which is essentially a safety stock, is regulated by the Fed, it is not affected by the DIs' decisions. Therefore, we only account for fit cash kept over and above this amount.

Dynamic Program (DP):

STAGE: Each period. There are T stages, $t = 1, 2, \ldots, T$.

STATE: The inventories of used cash and fit cash at the end of period t, (I_t^u, I_t^f).

DECISION: The quantity of used cash deposited to the Fed and the quantity of fit cash withdrawn from the Fed at the end of period t, (x_t^u, x_t^f).

OBJECTIVE: $\min_{I_T^u} G_T(I_T^u, 0)$

Cost Calculation:

Given $v_{t-1} = (I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$ (i.e., the state of the system and the decisions made at the end of Period $t-1$), three types of costs are incurred in Period t:

1. Holding cost (H_t)

   After the 3PLP transports the cash between the DI and the Fed, the inventory of used cash (resp., fit cash) at the beginning of Period t is $I_{t-1}^u - x_{t-1}^u$ (resp., $I_{t-1}^f + x_{t-1}^f$). At the end of period t, the inventory of used cash (resp., fit cash) is $I_t^u = I_{t-1}^u - x_{t-1}^u + d_t^u$ (resp., $I_t^f = I_{t-1}^f + x_{t-1}^f - d_t^f$). In our models, the duration of each period is one week; the week-to-week demands and deposits are deterministic, but non-stationary. The daily demand rate *within* each week is assumed to be constant. Thus, the holding cost of used cash (resp., fit cash) for Period t is calculated as the product of the per-bundle per-period holding cost and the average of the inventories of used cash (resp., fit cash) at the beginning and the end of Period t. Thus, the holding cost of the DI during Period t is:

$$H_t(v_{t-1}) = 0.5h[(I^u_{t-1} - x^u_{t-1}) + (I^u_{t-1} - x^u_{t-1} + d^u_t) + (I^f_{t-1} + x^f_{t-1}) + (I^f_{t-1} + x^f_{t-1} - d^f_t)].$$

2. Ordering cost (O_t)

   If, at the end of period $t-1$, the DI either deposits used cash or withdraws fit cash, the 3PLP charges a fixed cost s during Period t.

$$O_t(v_{t-1}) = \begin{cases} 0, & \text{if } x^u_{t-1} = x^f_{t-1} = 0 \\ s, & \text{otherwise.} \end{cases}$$

3. Cross-shipping cost (R_t)

   If, at the end of Period $t-1$, the DI both deposits used cash and withdraws fit cash, the Fed charges a cross-shipping fee during Period t. Recall from Section 3.1 that the per-bundle cross-shipping fee of e is charged on the minimum of the fit-cash withdrawn and the fit cash in the used-cash deposited. Since ρ is the percentage of fit cash in the used cash, we have

$$R_t(v_{t-1}) = e\min\{\rho x^u_{t-1}, x^f_{t-1}\}.$$

Thus, the total cost incurred during Period t is:

$$C_t(v_{t-1}) = H_t(v_{t-1}) + O_t(v_{t-1}) + R_t(v_{t-1}).$$

Note that given a solution $X = (x^u_0, x^f_0, x^u_1, x^f_1, \ldots, x^u_T, x^f_T)$ of BM, the value of v_{t-1} is determined. To simplify our exposition in the proofs of Theorems 3.2.1 and 3.2.3, given a solution

X of BM, we simply use $O_t(X)$ (resp., $R_t(X)$, $H_t(X)$, and $C_t(X)$) to represent $O_t(v_{t-1})$ (resp., $R_t(v_{t-1})$, $H_t(v_{t-1})$, and $C_t(v_{t-1})$).

It is easy to see that both the state space and the number of possible decision vectors of the DP defined above are exponential in the input size of an instance of BM. We now derive two important structural results for BM that can help us significantly reduce the state space of the DP, and consequently obtain a polynomial-time algorithm for BM.

3.2.2 Two Structural Results for BM

Theorem 3.2.1 identifies a characteristic of an optimal solution regarding the withdrawals of fit-cash: there exists an optimal solution in which fit cash is withdrawn only when the fit-cash inventory becomes zero. Theorem 3.2.3 establishes a property of used-cash deposits: there exists an optimal solution in which a DI will always deposit either none or all of its used-cash inventory.

Before we introduce our results, it is instructive to compare and contrast the management of inventory in our model with that in single-product lot-sizing problems and in joint-replenishment problems (e.g., Wagner and Whitin, 1958; Joneja, 1990). Used cash and fit cash of the same denomination can be considered as two classes of the same product, with one *partially convertible* (i.e., 75% convertible) into the other via fit-sorting. The costs of managing the inventories for these two classes are inter-dependent in a somewhat complicated manner. On the one hand, if used-cash is deposited and fit-cash is ordered together in the same period, then one realizes savings as there is only one transportation order as opposed to two if these two actions occur in two different periods. However, an additional cross-shipping penalty is incurred if both these actions occur together in the same period. Furthermore, the cross-shipping cost is proportional to the minimum of the volumes of the used-cash deposited and the fit-cash withdrawn. Clearly, there is a need to jointly manage the inventories of the two classes in BM. To further highlight our unique setting in the joint replenishment realm, note that the DI's transactions (of used- and fit-cash) with the Fed at the end of a period impact the inventories of the two classes in two opposite ways: the inventory of fit-cash increases and that of used-cash decreases after the transaction.

Theorem 3.2.1 *There exists an optimal solution, say X, of BM such that either $x_t^f = 0$ or $I_t^f = 0$ for $t = 1, 2, \ldots, T$.*

Proof: Consider an arbitrary solution X of BM and let t be the earliest period such that both $x_t^f > 0$ and $I_t^f > 0$. Next, let $t-k$ be the latest period before t such that $x_{t-k}^f > 0$ and $I_{t-k}^f = 0$, $1 \leq k \leq t$. The existence of such a period is guaranteed since we have $x_0^f > 0$ and $I_0^f = 0$. Thus, $t-k$ has at least one possible value, namely 0. It follows immediately from the definitions of $t-k$ and t that there is no withdrawal of fit cash by the DI (from the Fed) between these two periods. Thus, we have $x_{t-k}^f > I_t^f$.

Let $\delta = I_t^f$. We now define another solution $\bar{X}$ of BM as follows. Let $\bar{x}_i^u = x_i^u, i = 1, 2, \ldots, T$; $\bar{x}_{t-k}^f = x_{t-k}^f - \delta; \bar{x}_t^f = x_t^f + \delta; \bar{x}_i^f = x_i^f, i \in \{0, 1, \ldots, T\} \backslash \{t-k, t\}$. The only differences between $\bar{X}$ and X are the volumes of fit-cash withdrawals at the end of Periods $t-k$ and t. Thus, the difference between the total ordering costs (resp., cross-shipping costs) of these two solutions includes only the differences of the ordering costs (resp., cross-shipping costs) during Period $t-k+1$ and Period $t+1$. Hence,

$$O(\bar{X}) - O(X) = O_{t-k+1}(\bar{X}) - O_{t-k+1}(X) + O_{t+1}(\bar{X}) - O_{t+1}(X),$$

$$R(\bar{X}) - R(X) = R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) + R_{t+1}(\bar{X}) - R_{t+1}(X).$$

Since $x^f_{t-k} > 0, \bar{x}^f_{t-k} > 0$, we have $O_{t-k+1}(\bar{X}) - O_{t-k+1}(X) = s - s = 0$. Since $x^f_t > 0, \bar{x}^f_t > 0$, we have $O_{t+1}(\bar{X}) - O_{t+1}(X) = s - s = 0$. Thus, $O(\bar{X}) - O(X) = 0$. As compared to solution X, the total inventory of fit cash held by solution $\bar{X}$ between the end of Period $t-k$ and the end of Period t is less by an amount $\delta > 0$. Thus, $H(\bar{X}) - H(X) = -\delta hk$ and $\bar{I}^f_t = I^f_t - \delta = 0$. Hence, we have $C(\bar{X}) - C(X) = H(\bar{X}) - H(X) + O(\bar{X}) - O(X) + R(\bar{X}) - R(X) = -\delta hk + 0 + R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) + R_{t+1}(\bar{X}) - R_{t+1}(X)$. The possible values of $R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) = e(\min\{\rho x^u_{t-k}, \bar{x}^f_{t-k}\} - \min\{\rho x^u_{t-k}, x^f_{t-k}\})$ are as follows:

(A_1) If $\rho x^u_{t-k} \leq \bar{x}^f_{t-k}$, we have $R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) = 0$.

(A_2) If $\bar{x}^f_{t-k} < \rho x^u_{t-k} < x^f_{t-k}$, we have $R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) = e(\bar{x}^f_{t-k} - \rho x^u_{t-k}) > -e\delta$.

(A_3) If $\rho x^u_{t-k} \geq x^f_{t-k}$, we have $R_{t-k+1}(\bar{X}) - R_{t-k+1}(X) = -e\delta$.

Next, we consider the possible values of $R_{t+1}(\bar{X}) - R_{t+1}(X) = e(\min\{\rho x^u_t, \bar{x}^f_t\} - \min\{\rho x^u_t, x^f_t\})$:

(B_1) If $\rho x^u_t \leq x^f_t$, we have $R_{t+1}(\bar{X}) - R_{t+1}(X) = 0$.

(B_2) If $x^f_t < \rho x^u_t < \bar{x}^f_t$, we have $R_{t+1}(\bar{X}) - R_{t+1}(X) = e(\rho x^u_t - x^f_t) < e\delta$.

(B_3) If $\rho x^u_t \geq \bar{x}^f_t$, we have $R_{t+1}(\bar{X}) - R_{t+1}(X) = e\delta$.

Thus, we have nine possible combinations $(A_i, B_j), i = 1, 2, 3; j = 1, 2, 3$. For each of the five combinations $(A_1, B_1), (A_2, B_1), (A_3, B_1), (A_3, B_2)$, and (A_3, B_3), we have $R_{t-k+1}(\bar{X}) -$

$R_{t-k+1}(X) + R_{t+1}(\bar{X}) - R_{t+1}(X) \leq 0$. Since $-\delta hk < 0$, we have $C(\bar{X}) - C(X) < 0$. Thus, $\bar{X}$ is a better solution than X.

Let $\gamma = x_t^f$. We define another solution $\hat{X}$ of BM as follows: $\hat{x}_i^u = x_i^u, i = 1, 2, \ldots, T$; $\hat{x}_{t-k}^f = x_{t-k}^f + \gamma; \hat{x}_t^f = 0; \hat{x}_i^f = x_i^f, i \in \{0, 1, \ldots, T\} \backslash \{t-k, t\}$. Thus, $H(\hat{X}) - H(X) = \gamma hk$. Since $x_{t-k}^f > 0, \hat{x}_{t-k}^f > 0$, we have $O_{t-k+1}(\hat{X}) - O_{t-k+1}(X) = s - s = 0$. Since $x_t^f > 0, \hat{x}_t^f = 0$, we have $O_{t+1}(\hat{X}) - O_{t+1}(X) \leq s - s = 0$. Thus, we have $C(\hat{X}) - C(X) = H(\hat{X}) - H(X) + O(\hat{X}) - O(X) + R(\hat{X}) - R(X) \leq \gamma hk + 0 + R_{t-k+1}(\hat{X}) - R_{t-k+1}(X) + R_{t+1}(\hat{X}) - R_{t+1}(X)$. Under each of the four combinations $(A_1, B_2), (A_1, B_3), (A_2, B_2)$, and (A_2, B_3), we have $R_{t-k+1}(\hat{X}) - R_{t-k+1}(X) = 0$, $R_{t+1}(\hat{X}) - R_{t+1}(X) = -e\gamma$. Thus, $C(\hat{X}) - C(X) \leq \gamma hk - e\gamma$. We now investigate these 4 combinations individually.

(i) (A_1, B_2): When $\rho x_{t-k}^u \leq \bar{x}_{t-k}^f$ and $x_t^f < \rho x_t^u < \bar{x}_t^f$, we have $C(\bar{X}) - C(X) = -\delta hk + e(\rho x_t^u - x_t^f)$. If $C(\bar{X}) - C(X) < 0$, then $\bar{X}$ is a better solution than X. If $C(\bar{X}) - C(X) \geq 0$, then we have $-\delta hk + e(\rho x_t^u - x_t^f) \geq 0$. Since $0 < \rho x_t^u - x_t^f < \delta$, we have $e > hk$. Then, $C(\hat{X}) - C(X) \leq \gamma hk - e\gamma = \gamma(hk - e) < 0$. Therefore, $\hat{X}$ is a better solution than X.

(ii) (A_1, B_3): When $\rho x_{t-k}^u \leq \bar{x}_{t-k}^f$ and $\rho x_t^u \geq \bar{x}_t^f$, we have $C(\bar{X}) - C(X) = -\delta hk + e\delta$. If $C(\bar{X}) - C(X) < 0$, then $\bar{X}$ is a better solution than X. If $C(\bar{X}) - C(X) \geq 0$, then we have $-\delta hk + e\delta \geq 0$. Thus, $e \geq hk$ and, therefore, $C(\hat{X}) - C(X) \leq \gamma hk - e\gamma = \gamma(hk - e) \leq 0$. Consequently, $\hat{X}$ is at least as good as X.

(iii) (A_2, B_2): When $\bar{x}_{t-k}^f < \rho x_{t-k}^u < x_{t-k}^f$ and $x_t^f < \rho x_t^u < \bar{x}_t^f$, we have $C(\bar{X}) - C(X) = -\delta hk + e(\bar{x}_{t-k}^f - \rho x_{t-k}^u) + e(\rho x_t^u - x_t^f)$. If $C(\bar{X}) - C(X) < 0$, then $\bar{X}$ is a better solution than X. If $C(\bar{X})-C(X)\geq 0$, then we have $-\delta hk+e(\bar{x}^f_{t-k}-\rho x^u_{t-k})+e(\rho x^u_t-x^f_t)\geq 0$. Since $0<(\bar{x}^f_{t-k}-\rho x^u_{t-k})+(\rho x^u_t-x^f_t)<\delta$, we have $e>hk$. Therefore, $C(\hat{X})-C(X)\leq \gamma hk-e\gamma=\gamma(hk-e)<0$. Again, $\hat{X}$ is a better solution than X.

(iv) (A_2,B_3): When $\bar{x}^f_{t-k}<\rho x^u_{t-k}<x^f_{t-k}$ and $\rho x^u_t\geq \bar{x}^f_t$, we have $C(\bar{X})-C(X)=-\delta hk+e(\bar{x}^f_{t-k}-\rho x^u_{t-k})+e\delta$. If $C(\bar{X})-C(X)<0$, $\bar{X}$ is a better solution than X. Otherwise, if $C(\bar{X})-C(X)\geq 0$, then we have $-\delta hk+e(\bar{x}^f_{t-k}-\rho x^u_{t-k})+e\delta\geq 0$. Since $0<\bar{x}^f_{t-k}-\rho x^u_{t-k}+\delta<\delta$, we have $e>hk$. Therefore, $C(\hat{X})-C(X)\leq\gamma hk-e\gamma=\gamma(hk-e)<0$ and, hence, $\hat{X}$ is a better solution than X.

Thus, if in a solution X of BM, there exists some t such that $x^f_t>0$ and $I^f_t>0$, we can obtain another solution (either $\bar{X}$ of $\hat{X}$) which satisfies either $x^f_t=0$ or $I^f_t=0$, without increasing the total cost. Finally, if there exist multiple periods that satisfy $x^f_t>0$ and $I^f_t>0$, the argument above can be repeated – starting from the earliest such period – for each of these periods, to obtain an optimal solution which satisfies either $x^f_t=0$ or $I^f_t=0$ for $t=1,2,\ldots T$. The result follows. ■

Theorem 3.2.1 immediately implies the following result.

Corollary 3.2.2 *There exists an optimal solution X of BM such that when $x^f_t>0$, we have $x^f_t=\sum_{j=1}^k d^f_{t+j}$, for some $k\in\{1,2,\ldots,T-t\}$. Also, $I^f_t=-d^f_t+\sum_{j=0}^k d^f_{t+j}$ for some $k\in\{0,1,\ldots,T-t\}$. Thus, I^f_t has $(T-t+1)$ possible values.*

While Theorem 3.2.1 examined the decisions on the withdrawal of fit cash from the Fed, the following result analyzes the decisions on the deposits of used cash to the Fed.

Theorem 3.2.3 *There exists an optimal solution, say X, of BM such that either $x_t^u = 0$ or $x_t^u = I_t^u$ for every $t = 1, 2, \ldots, T$.*

Proof: Let X be an arbitrary solution of BM and t be the latest period with $0 < x_t^u < I_t^u$. Then, let $t+k$ be the first period after t such that $x_{t+k}^u = I_{t+k}^u$, $1 \leq k \leq T-t$. Since $x_T^u = I_T^u > 0$, note that $t+k$ has at least one possible value, namely T. Also, we have $x_{t+k}^u > I_t^u - x_t^u$ since the definitions of t and $t+k$ imply that no used-cash deposit occurs between these two periods.

Let $\delta = I_t^u - x_t^u > 0$. We now define another solution of BM, $\bar{X}$. This solution is the same as X, except for the values of the deposits of used cash at the end of Period t and Period $t+k$. Let $\bar{x}_t^u = I_t^u = x_t^u + \delta, \bar{x}_{t+k}^u = x_{t+k}^u - \delta$. Since $x_t^u > 0, \bar{x}_t^u > 0$, and $x_{t+k}^u > 0, \bar{x}_{t+k}^u > 0$, we have $O_{t+1}(\bar{X}) - O_{t+1}(X) = s - s = 0$ and $O_{t+k+1}(\bar{X}) - O_{t+k+1}(X) = s - s = 0$. Thus, $C(\bar{X}) - C(X) = -\delta hk + R_{t+1}(\bar{X}) - R_{t+1}(X) + R_{t+k+1}(\bar{X}) - R_{t+k+1}(X)$. Also, $R_{t+1}(\bar{X}) - R_{t+1}(X) = e(\min\{\rho x_t^u + \rho\delta, x_t^f\} - \min\{\rho x_t^u, x_t^f\}) \leq e\rho\delta$, and $R_{t+k+1}(\bar{X}) - R_{t+k+1}(X) = e(\min\{\rho x_{t+k}^u - \rho\delta, x_{t+k}^f\} - \min\{\rho x_{t+k}^u, x_{t+k}^f\}) \leq 0$. It follows that $C(\bar{X}) - C(X) \leq -\delta hk + e\rho\delta = (-hk + e\rho)\delta$.

If $hk > e\rho$, we have $C(\bar{X}) < C(X)$. Thus, $\bar{X}$ is a better solution than X.

Otherwise, if $hk \leq e\rho$, then there are three possibilities:

1. If $\rho x_t^u \geq x_t^f$, then, $R_{t+1}(\bar{X}) - R_{t+1}(X) = 0$. Since $R_{t+k+1}(\bar{X}) - R_{t+k+1}(X) \leq 0$, we have $C(\bar{X}) - C(X) \leq 0 - \delta hk < 0$, which implies $\bar{X}$ is a better solution than X.

2. If $\rho x_{t+k}^u \leq x_{t+k}^f$, then $R_{t+k+1}(\bar{X}) - R_{t+k+1}(X) = -e\rho\delta$. Since $R_{t+1}(\bar{X}) - R_{t+1}(X) \leq e\rho\delta$,

we have $C(\bar{X}) - C(X) \leq e\rho\delta - e\rho\delta - \delta hk < 0$, which implies $\bar{X}$ is a better solution than X.

3. If $\rho x_t^u < x_t^f$ and $\rho x_{t+k}^u > x_{t+k}^f$, we let $\gamma = x_t^u$, and consider another solution $\hat{X}$ of BM, defined as follows: $\hat{x}_i^f = x_i^f, i = 1, 2, \ldots, T$; $\hat{x}_t^u = 0; \hat{x}_{t+k}^u = x_{t+k}^u + \gamma; \hat{x}_i^u = x_i^u, i \in \{1, 2, \ldots, T\} \backslash \{t, t+k\}$. Since $x_t^u > 0, \hat{x}_t^u = 0$, we have $O_{t+1}(\hat{X}) - O_{t+1}(X) \leq s - s = 0$. Since $x_{t+k}^u > 0, \hat{x}_{t+k}^u > 0$, we have $O_{t+k+1}(\hat{X}) - O_{t+k+1}(X) = s - s = 0$. Thus, we have $C(\hat{X}) - C(X) \leq \gamma hk + R_{t+1}(\hat{X}) - R_{t+1}(X) + R_{t+k+1}(\hat{X}) - R_{t+k+1}(X)$. We have $R_{t+1}(\hat{X}) - R_{t+1}(X) = -e\rho\gamma$, $R_{t+k+1}(\hat{X}) - R_{t+k+1}(X) = 0$. Therefore, $C(\hat{X}) - C(X) \leq \gamma hk - e\rho\gamma = (hk - e\rho)\gamma$. Since $hk \leq e\rho$, we have $C(\hat{X}) - C(X) \leq 0$, which implies $\hat{X}$ is at least as good as X.

To summarize, if in a solution X of BM, there exists some t such that $0 < x_t^u < I_t^u$, we can always find another solution which satisfies either $x_t^u = 0$ or $x_t^u = I_t^u$, without increasing the total cost. If there exist multiple periods with $0 < x_t^u < I_t^u$, then the argument above can be repeated for each such period, starting from the latest of these periods. This concludes the proof. ∎

The following result follows immediately from Theorem 3.2.3.

Corollary 3.2.4 *There exists an optimal solution of BM for which* $I_t^u = \sum_{j=1}^{k} d_{t+1-j}^u$ *for some* $k \in \{1, 2, \ldots, t\}$. *Thus,* I_t^u *has* t *possible values.*

Together, Theorems 3.2.1 and 3.2.3 allow us to significantly reduce the enumeration of the states when searching for an optimal solution of BM. We now introduce the forward recursion function of DP.

3.2.3 Forward Recursion Function of DP and its Computational Complexity

Recall from Section 3.2.1 that (i) $G_t(I_t^u, I_t^f)$ is the minimum cost incurred during periods 1 through t, given that, at the end of Period t, the inventory of used cash is I_t^u and the inventory of fit cash is I_t^f and (ii) $C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$ is the total cost incurred during Period t.

From Theorem 3.2.1 and the relationship $I_t^f = I_{t-1}^f + x_{t-1}^f - d_t^f$, to have a fit-cash inventory of I_t^f at the end of Period t, one of the following must hold at the end of Period $t-1$: (i) We have fit-cash inventory of 0 and order $I_t^f + d_t^f$ fit-cash from the Fed; i.e., $I_{t-1}^f = 0$ and $x_{t-1}^f = I_t^f + d_t^f$ or (ii) We have fit-cash inventory of $I_t^f + d_t^f$ and do not place an order for fit-cash from the Fed; i.e., $I_{t-1}^f = I_t^f + d_t^f, x_{t-1}^f = 0$. Since $I_0^u = I_0^f = x_0^u = 0, x_0^f > 0$, we have $I_1^u = d_1^u$. Thus, for $t = 1$, the forward recursion function is:

$$G_1(d_1^u, I_1^f) = C_1(0, 0, 0, I_1^f + d_1^f) \tag{3.1}$$

For $2 \leq t \leq T$, the forward recursions have the following forms, depending on the value of I_t^u:

1. If $I_t^u = d_t^u$, then the used-cash inventory is 0 at the beginning of Period t, which in turn implies that, at the end of Period $t-1$, we deposit all the used-cash inventory to the Fed (i.e., $x_{t-1}^u = I_{t-1}^u$). Thus, in this case, the forward recursion function is:
$$G_t(d_t^u, I_t^f) = \min_{I_{t-1}^u} \begin{cases} G_{t-1}(I_{t-1}^u, I_t^f + d_t^f) + C_t(I_{t-1}^u, I_t^f + d_t^f, I_{t-1}^u, 0), \\ G_{t-1}(I_{t-1}^u, 0) + C_t(I_{t-1}^u, 0, I_{t-1}^u, I_t^f + d_t^f) \end{cases} \tag{3.2}$$

2. If $I_t^u > d_t^u$, then the used-cash inventory is positive at the beginning of Period t. From Theorem 3.2.3, we have $x_{t-1}^u = 0$. Consequently, we have $I_{t-1}^u = I_t^u - d_t^u$ and the

forward recursion function is:

$$G_t(I_t^u, I_t^f) = \min \begin{cases} G_{t-1}(I_t^u - d_t^u, I_t^f + d_t^f) + C_t(I_t^u - d_t^u, I_t^f + d_t^f, 0, 0), \\ G_{t-1}(I_t^u - d_t^u, 0) + C_t(I_t^u - d_t^u, 0, 0, I_t^f + d_t^f) \end{cases} \tag{3.3}$$

Theorem 3.2.5 *The computational complexity of DP is $O(T^3)$.*

Proof: (i) For $t = 1$, the forward recursion function is Equation 3.1. From Corollary 3.2.2, I_1^f has T possible values. Thus, there are T possible values of $C_1(0, 0, 0, I_1^f + d_1^f)$.

(ii) When $2 \le t \le T$, from Corollary 3.2.2, I_t^f has $(T - t + 1)$ possible values. Also, the values of I_t^f and I_t^u are independent of each other. We have two possibilities:

(1) The forward recursion function is Equation 3.2. Then, from Corollary 3.2.4, I_{t-1}^u has $t - 1$ possible values. Thus, for each state defined by Equation 3.2, there are $2(t - 1)$ possible combinations of $G_{t-1}(I_{t-1}^u, I_{t-1}^f) + C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$. Since there are $(T - t + 1)$ states defined by Equation 3.2, the total number of possible combinations of $G_{t-1}(I_{t-1}^u, I_{t-1}^f) + C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$ for over all these states is $2(t-1)(T+1-t)$.

(2) The forward recursion function is Equation 3.3. For each state defined by Equation 3.3, there are 2 possible combinations of $G_{t-1}(I_{t-1}^u, I_{t-1}^f) + C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$. From Corollary 3.2.4, there are t possible values of I_t^u. Among them, $(t - 1)$ values satisfy $I_t^u > d_t^u$. Thus, the number of states defined by Equation 3.3 is $(t - 1)(T + 1 - t)$. Consequently, there are a total of $2(t - 1)(T + 1 - t)$ possible combinations of $G_{t-1}(I_{t-1}^u, I_{t-1}^f) + C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$ over all the states defined by Equation 3.3.

Over all values of t (i.e., $t = 1, 2, \ldots, T$), the total number, $\xi(T)$, of possible combinations of $G_{t-1}(I_{t-1}^u,\ I_{t-1}^f) + C_t(I_{t-1}^u, I_{t-1}^f, x_{t-1}^u, x_{t-1}^f)$ that the DP evaluates for a T-period instance of

BM is, $T+\sum_{t=2}^{T}[2(t-1)(T+1-t)+2(t-1)(T+1-t)] = \frac{2}{3}T^3+\frac{1}{3}T$. Since each combination can be evaluated in time $O(1)$, the computational complexity of DP is $O(T^3)$. ∎

The primary implication of Theorem 3.2.5 is that if a DI decides not to implement fit-sorting (and, therefore, also cannot exploit the custodial inventory option), then its optimal inventory decisions can be obtained efficiently.

3.2.4 A Generalized BM (BM^g)

The original BM (referred to here as BM^o, to avoid confusion) has the following assumptions: $I_0^f = 0$, $I_0^u = 0$, $I_T^f = 0$, $x_T^u = I_T^u$, $d_t^f > 0$, $d_t^u > 0$, $t = 1, 2, \ldots, T$. However, in practice, the DI may want to have some extra fit-cash inventory above the minimum amount required by the Fed (see Section 3.2.1) at the end of a planning horizon, i.e., $I_T^f > 0$. Consequently, fit-cash inventory can be positive at the beginning of the next planning horizon, i.e., $I_0^f > 0$. For practical reasons, we assume a DI's purpose of carrying fit-cash inventory over planning horizons is to satisfy customers' demand. Thus, we assume that the DI will not deposit its initial inventory of fit-cash back to the Fed. Similarly, we only consider the non-trivial case in which the DI needs to withdraw a positive amount of fit-cash during the planning horizon, i.e., $I_0^f < \sum_{t=1}^{T} d_t^f + I_T^f$. In this section, we study a generalized version of BM (BM^g) in which the initial and ending inventories of both fit cash and used cash are non-negative, i.e., $I_0^f \geq 0$, $I_0^u \geq 0$, $I_T^f \geq 0$, and $I_T^u \geq 0$. We first describe an intermediate problem (referred to as BM^c) which is derived from BM^g, and shares some structural similarities with BM^o. Next, we prove that BM^g and BM^c share the same optimal solution. Finally, we show that BM^c can be solved in polynomial time.

Given an instance of BM^g with $\hat{d}_t^f > 0$, $\hat{d}_t^u > 0$, $t = 1, 2, \ldots, T$, $\hat{I}_0^f \geq 0$, $\hat{I}_0^u \geq 0$, $\hat{I}_T^f \geq 0$, we now describe the procedure to convert it to an instance of BM^c: First, we copy the fit-cash demand of the first $T-1$ periods from BM^g to BM^c, and set the fit-cash demand of the last period in BM^c as the sum of the fit-cash demand of the last period in BM^g and the ending inventory of fit-cash, i.e., $d_T^f = \hat{d}_T^f + \hat{I}_T^f$. Second, we determine the value of t_1, the latest period until which the DI does not have to withdraw fit-cash from the Fed and, instead, use initial inventory to satisfy customers' demand. The value of t_1 is determined as follows: if $\hat{I}_0^f < d_1^f$, then $t_1 = 0$; otherwise, t_1 is defined by $\sum_{t=1}^{t_1} d_t^f \leq \hat{I}_0^f < \sum_{t=1}^{t_1+1} d_t^f$. Finally, we make the following updates: $I_0^f = 0$, $I_T^f = 0$; $d_t^f = 0$, $t = 1, 2, \ldots, t_1$; $d_{t_1+1}^f = \sum_{t=1}^{t_1+1} \hat{d}_t^f - \hat{I}_0^f$; $d_t^f = \hat{d}_t^f, t_1 + 2 \leq t \leq T-1$; $d_T^f = \hat{d}_T^f + \hat{I}_T^f$. Note that during the transformation, we only alter the demands of fit-cash and the initial and ending inventories of fit-cash. The used-cash deposits in BM^c are the same as in BM^g.

The following two claims characterize the relationship between the solutions of BM^g and BM^c from two aspects: feasibility and optimality.

Claim 3.2.1 *Each feasible solution to an instance of BM^g is a feasible solution to the corresponding instance of BM^c, and vice-versa.*

Proof: A feasible solution to an instance of BM^g, X, satisfies the following conditions:

$$\hat{I}_t^f = \hat{I}_0^f + \sum_{i=0}^{t-1} x_i^f - \sum_{i=1}^{t} \hat{d}_i^f \geq 0, 1 \leq t \leq T-1, \tag{3.4}$$

$$\hat{I}_0^f + \sum_{i=0}^{T-1} x_i^f - \sum_{i=1}^{T} \hat{d}_i^f - \hat{I}_T^f = 0, \tag{3.5}$$

$$x_t^u \leq \hat{I}_t^u = \sum_{i=1}^{t} \hat{d}_i^u - \sum_{i=0}^{t-1} x_i^u, 1 \leq t \leq T, \tag{3.6}$$

$$x_i^f \geq 0, x_i^u \geq 0, i = 0, 1, \ldots, T. \tag{3.7}$$

Similarly, the feasibility conditions to an instance of the corresponding BM^c are:

$$I_t^f = \sum_{i=0}^{t-1} x_i^f - \sum_{i=1}^{t} d_i^f \geq 0, 1 \leq t \leq T-1, \tag{3.8}$$

$$\sum_{i=0}^{T-1} x_i^f - \sum_{i=1}^{T} d_i^f = 0, \tag{3.9}$$

$$x_t^u \leq I_t^u = \sum_{i=1}^{t} d_i^u - \sum_{i=0}^{t-1} x_i^u, 1 \leq t \leq T, \tag{3.10}$$

$$x_i^f \geq 0, x_i^u \geq 0, i = 0, 1, \ldots, T. \tag{3.11}$$

$\Longrightarrow$ Assume $\hat{X}$ is a feasible solution to an instance of BM^g. Then $\hat{X}$ satisfies Constraints 3.4 - 3.7. Since $d_i^u = \hat{d}_i^u, i = 1, 2, \ldots, T$, it is clear that $\hat{X}$ satisfies Constraints 3.10 and 3.11. For $1 \leq t \leq t_1$, from the definition of t_1, we have $\sum_{i=1}^{t} d_i^f = 0$. From Constraint 3.7, we have $\hat{x}_i^f \geq 0, i = 0, 1, \ldots, T$. Thus, Constraint 3.8 holds for $1 \leq t \leq t_1$. When $t = t_1 + 1$, we have $-\sum_{i=1}^{t_1+1} d_i^f = 0 - d_{t_1+1}^f = \hat{I}_0^f - \sum_{i=1}^{t_1+1} \hat{d}_i^f$. For $t_1 + 2 \leq t \leq T-1$, since $\hat{d}_t^f = d_t^f$, we have $-\sum_{i=1}^{t} d_i^f = \hat{I}_0^f - \sum_{i=1}^{t} \hat{d}_i^f$. Since $\hat{X}$ satisfies Constraint 3.4, it also satisfies Constraint 3.8 for $t_1 + 1 \leq t \leq T-1$. Finally, for $t = T$, we have $-\sum_{i=1}^{T} d_i^f = -\sum_{i=1}^{T-1} d_i^f - d_T^f = (\hat{I}_0^f - \sum_{i=1}^{T-1} \hat{d}_i^f) - (\hat{d}_T^f + \hat{I}_T^f) = \hat{I}_0^f - \sum_{i=1}^{T} \hat{d}_i^f - \hat{I}_T^f$. Since Constraint 3.5 holds, $\hat{X}$ satisfies Constraint 3.9. Therefore, $\hat{X}$ satisfies Constraints 3.8-3.11.

$\Longleftarrow$ Assume $\bar{X}$ is a feasible solution to an instance of BM^c. Then $\bar{X}$ satisfies Constraints 3.8-3.11. Since $d_i^u = \hat{d}_i^u, i = 1, 2, \ldots, T$, Constraints 3.6 and 3.7 are satisfied by $\bar{X}$. For $1 \leq t \leq t_1$, from the definition of t_1, we have $\hat{I}_0^f - \sum_{i=1}^{t} \hat{d}_i^f \geq 0$ and $\sum_{i=1}^{t} d_i^f = 0$. Therefore, since $\bar{X}$ satisfies Constraint 3.8, it satisfies Constraint 3.4 for $1 \leq t \leq t_1$. For $t_1 + 1 \leq t \leq T$, the argument is similar to that in the first part of the proof; we avoid repeating it for brevity.

Therefore, $\bar{X}$ is also a feasible solution to the corresponding instance of BM^g. The result follows. $\square$

While Claim 3.2.1 shows that BM^g and BM^c share the same feasible set, the claim below analyzes the difference between the total costs incurred in these two instances for the same feasible solution.

Claim 3.2.2 *Assume X is a feasible solution to an instance of BM^g. Let $C^g(X)$ (resp., $C^c(X)$) represent the total cost corresponding to X in BM^g (resp., the corresponding instance of BM^c). Then, we have $C^g(X) - C^c(X) = \frac{1}{2}h[(2t_1+1)\hat{I}_0^f + \hat{I}_T^f - 2\sum_{t=1}^{t_1}\sum_{i=1}^{t}\hat{d}_i^f]$, a quantity that is independent of X.*

Proof: For X, the ordering cost (resp., cross-shipping cost) incurred in BM^g and the corresponding instance of BM^c is the same. Thus, the only difference between $C^g(X)$ and $C^c(X)$ is the holding cost for fit cash. Recall from Section 3.2.1 that the holding cost for fit-cash during Period t is $\frac{1}{2}h(I_{t-1}^f + x_{t-1}^f + I_t^f)$. Thus, we have

$$\begin{aligned} C^g(X) - C^c(X) &= \frac{1}{2}h\sum_{t=1}^{T}[(\hat{I}_{t-1}^f + x_{t-1}^f + \hat{I}_t^f) - (I_{t-1}^f + x_{t-1}^f + I_t^f)] \\ &= \frac{1}{2}h\sum_{t=1}^{T}(\hat{I}_{t-1}^f - I_{t-1}^f) + \frac{1}{2}h\sum_{t=1}^{T}(\hat{I}_t^f - I_t^f) \\ &= \frac{1}{2}h(\hat{I}_0^f + \hat{I}_T^f) + h\sum_{t=1}^{T-1}(\hat{I}_t^f - I_t^f). \end{aligned}$$

Since $\hat{I}_t^f = \hat{I}_0^f + \sum_{i=0}^{t-1}x_i^f - \sum_{i=1}^{t}\hat{d}_i^f$, $I_t^f = I_0^f + \sum_{i=0}^{t-1}x_i^f - \sum_{i=1}^{t}d_i^f$, $t = 1, 2, \ldots, T-1$, we have $\hat{I}_t^f - I_t^f = \hat{I}_0^f + \sum_{i=0}^{t-1}x_i^f - \sum_{i=1}^{t}\hat{d}_i^f - (I_0^f + \sum_{i=0}^{t-1}x_i^f - \sum_{i=1}^{t}d_i^f) = \hat{I}_0^f - \sum_{i=1}^{t}\hat{d}_i^f + \sum_{i=1}^{t}d_i^f$, $t = 1, 2, \ldots, T-1$.

For $1 \le t \le t_1$, we have $\hat{I}^f_t - I^f_t = \hat{I}^f_0 - \sum_{i=1}^t \hat{d}^f_i$. For $t = t_1 + 1$, we have

$$\begin{aligned} \hat{I}^f_{t_1+1} - I^f_{t_1+1} &= \hat{I}^f_0 - \sum_{i=1}^{t_1+1} \hat{d}^f_i + \sum_{i=1}^{t_1+1} d^f_i \\ &= \hat{I}^f_0 - \sum_{i=1}^{t_1+1} \hat{d}^f_i + \sum_{i=1}^{t_1} d^f_i + d^f_{t_1+1} \\ &= \hat{I}^f_0 - \sum_{i=1}^{t_1+1} \hat{d}^f_i + 0 + (\sum_{i=1}^{t_1+1} \hat{d}^f_i - \hat{I}^f_0) = 0. \end{aligned}$$

For $t_1 + 2 \le t \le T - 1$, since $\hat{d}^f_t = d^f_t$ and $\hat{I}^f_{t_1+1} = I^f_{t_1+1}$, we have $\hat{I}^f_t - I^f_t = 0$.

Combining the above three results, we have

$$C^g(X) - C^c(X) = \frac{1}{2}h(\hat{I}^f_0 + \hat{I}^f_T) + h\sum_{t=1}^{t_1}(\hat{I}^f_0 - \sum_{i=1}^{t} \hat{d}^f_i) = \frac{1}{2}h[(2t_1+1)\hat{I}^f_0 + \hat{I}^f_T - 2\sum_{t=1}^{t_1}\sum_{i=1}^{t} \hat{d}^f_i],$$

a quantity that is independent of X. □

The following theorem follows immediately from the above two claims.

Theorem 3.2.6 *Every optimal solution to an instance of BM^g is an optimal solution to the corresponding instance of BM^c, and vice-versa.*

Thus, to solve an instance of BM^g, we can solve the corresponding instance of BM^c. Next, we compare BM^c with BM^o and derive some structural properties of BM^c. In both BM^c and BM^o, we have $I^f_0 = 0$, $I^f_T = 0$. However, in BM^c, the fit-cash demands of the first t_1 periods are all zero. This violates the assumption of $d^f_t > 0$, $t = 1, 2, \ldots, T$, which is required for BM^o. However, the following result shows that the property described in Theorem 3.2.1 (Section 3.2.2) continues to hold for BM^c.

Theorem 3.2.7 *There exists an optimal solution, say X, of BM^c such that either $x^f_i = 0$ or $I^f_i = 0$ for $i = 0, 1, \ldots, T$.*

Proof: Consider an arbitrary solution X of BM^c and let t_0 be the first period the DI withdraws fit-cash from the Fed. We first observe that $0 \leq t_0 \leq t_1$. Suppose $t_0 \geq t_1 + 1$, then we have $x_i^f = 0$ for $0 \leq i \leq t_1$, and $I_{t_1+1}^f = I_0^f + \sum_{i=0}^{t_1} x_i^f - \sum_{i=1}^{t_1+1} d_i^f = -d_1^f < 0$, which contradicts the assumption that X is an optimal solution of BM^c. Hence, $0 \leq t_0 \leq t_1$. It is easy to see that either $x_i^f = 0$ or $I_i^f = 0$ for $i = 0, 1, \ldots, t_0$.

Next we show that either $x_i^f = 0$ or $I_i^f = 0$ for $i = t_0 + 1, t_0 + 2, \ldots, T$. Let t be the earliest period such that both $x_t^f > 0$ and $I_t^f > 0$. Obviously, $t \geq t_0 + 1$. Let $t - k$ be the latest period before t such that $x_{t-k}^f > 0$ and $I_{t-k}^f = 0$, $1 \leq k \leq t$. The existence of such a period is guaranteed since we have $x_{t_0}^f > 0$ and $I_{t_0}^f = 0$. Thus, $t - k$ has at least one possible value, namely t_0. It follows immediately from the definitions of $t - k$ and t that there is no withdrawal of fit cash by the DI (from the Fed) between these two periods. Thus, we have $x_{t-k}^f > I_t^f$. The remaining part of the proof is similar to that in the proof of Theorem 3.2.1. We, therefore, avoid repeating it for brevity. ■

Theorem 3.2.7 immediately implies the following result.

Corollary 3.2.8 *There exists an optimal solution X of BM^c such that*

(i) We have $x_{t_0}^f > 0$ for some $t_0 \in \{0, 1, \ldots, t_1\}$ and $x_t^f = 0 \; \forall t \in \{0, 1, \ldots, t_1\} \backslash \{t_0\}$. Also, $x_{t_0}^f = \sum_{j=1}^{k} d_{t_1+j}^f$ for some $k \in \{1, 2, \ldots, T - t_1\}$. For $0 \leq t \leq t_1$, we have $I_t^f = 0$ or $I_t^f = \sum_{j=1}^{k} d_{t_1+j}^f$ for some $k \in \{1, 2, \ldots, T - t_1\}$. I_t^f has $(T - t_1 + 1)$ possible values.

(ii) For $t_1 + 1 \leq t \leq T$, if $x_t^f > 0$, we have $x_t^f = \sum_{j=1}^{k} d_{t+j}^f$ for some $k \in \{1, 2, \ldots, T - t\}$. Also, $I_t^f = -d_t^f + \sum_{j=0}^{k} d_{t+j}^f$ for some $k \in \{0, 1, \ldots, T - t\}$. Thus, I_t^f has $(T - t + 1)$ possible values.

In the original BM (BM^o), we require DI to deposit all the used-cash inventory at the end of planning horizon, i.e., $x^u_T = I^u_T$. However, in BM^c, x^u_{T-1} is last decision made by DI for used-cash deposit. Next through three claims, we show in Theorem 3.2.9 that the property shown in Theorem 3.2.3 still holds for BM^c.

Claim 3.2.3 *There exists an optimal solution, say X, of BM^c such that either $x^u_{T-1} = 0$ or $x^u_{T-1} = I^u_{T-1}$.*

Proof: Let X be an solution which satisfies $0 < x^u_{T-1} < I^u_{T-1}$. Let $\delta = I^u_{T-1} - x^u_{T-1}$. We define another solution of BM^c, $\bar{X}$. Except for $\bar{x}^u_{T-1} = x^u_{T-1} + \delta$, all the other decisions of $\bar{X}$ share the same value as in X. Thus, $O_T(\bar{X}) - O_T(X) = 0$, $R_T(\bar{X}) - R_T(X) = e(min\{\rho x^u_{T-1} + \rho\delta, x^f_{T-1}\} - min\{\rho x^u_{T-1}, x^f_{T-1}\}) \leq e\rho\delta$. It follows that $C(\bar{X}) - C(X) \leq -\delta h + e\rho\delta$.

If $h > e\rho$, then $C(\bar{X}) - C(X) < 0$. Otherwise, we have $h \leq e\rho$. If $\rho x^u_{T-1} \geq x^f_{T-1}$, then $R_T(\bar{X}) - R_T(X) = 0$, $C(\bar{X}) - C(X) < 0$. If $\rho x^u_{T-1} < x^f_{T-1}$, we consider another solution $\hat{X}$, which is the same as X, except for $\hat{x}^u_{T-1} = 0$. Thus, $O_T(\hat{X}) - O_T(X) \leq 0$, $R_T(\hat{X}) - R_T(X) = 0 - e\rho x^u_{T-1}$. It follows that $C(\bar{X}) - C(X) \leq x^u_{T-1}(h - e\rho) \leq 0$. The results follow. □

Using the similar method, we can prove the following claim.

Claim 3.2.4 *There exists an optimal solution, say X, of BM^c such that if $x^u_{T-1} = 0$, then either $x^u_t = 0$ or $x^u_t = I^u_t$ for every $t = 1, 2, \ldots, T$.*

Proof: Let X be an solution which satisfies $x^u_{T-1} = 0$ and $0 < x^u_t < I^u_t$, $0 \leq t < T-1$. By using a similar method as in the proof of Claim 3.2.3, we can obtain a solution no worse than X by only changing x^u_t to either 0 or I^u_t. We omit the details here for brevity. □

The proof of following claim is similar to that of Theorem 3.2.3.

Claim 3.2.5 *There exists an optimal solution, say X, of BM^c such that if $x^u_{T-1} = I^u_{T-1}$, then either $x^u_t = 0$ or $x^u_t = I^u_t$ for every $t = 1, 2, \ldots, T$.*

Combining Claims 3.2.3, 3.2.4, 3.2.5, we have the following theorem.

Theorem 3.2.9 *There exists an optimal solution, say X, of BM^c such that either $x^u_i = 0$ or $x^u_i = I^u_i$ for $i = 0, 1, \ldots, T-1$.*

We can now use a forward recursion similar to that described in Section 3.2.3 to optimize BM^c. The computational complexity of the corresponding DP for BM^c remains $O(T^3)$. Our next task is to discuss the scenario when the DI implements the Fed's policy.

3.3 The Reuse Model (RM): Inventory Decisions under the Fed's Recirculation Policy

Under the (local) Reuse Model (RM), a DI can recirculate cash by using the fit-sorting and custodial inventory options. We first describe the model in Section 3.3.1. Then, in Section 3.3.2, we show that the decision problem corresponding to RM is NP-complete and establish a structural property for RM. Finally, in Section 3.3.3, we provide a mixed-integer programming formulation of RM that is used in Section 3.4.

3.3.1 Problem Description

Structurally, RM is a much more complicated model than BM. There are four additional parameters (defined below) in RM. Moreover, for each period, RM has four additional decision variables and two more types of inventories. Under RM, the **Timeline of the Activities** conducted by the DI and the 3PLP is significantly different than that in BM and is described below.

- At the end of Period t, the DI takes the following actions (in the given order) (i) counts the inventories of used cash, unfit cash, fit cash, and custodial inventory, (ii) decides the quantity of used cash to fit-sort. As mentioned earlier in Section 3.1, after fit-sorting (if any), about 75% of the fit-sorted used cash is fit cash and the remaining 25% is unfit cash, (iii) decides the quantity of fit-sorted cash to be placed in custodial inventory. Thus, the remainder of the fit-sorted cash will be directly placed in the DI's fit-cash inventory, (iv) decides the quantities of used cash and unfit cash to be deposited to the Fed, the quantity of fit cash to be withdrawn from the Fed, and the quantity of fit cash to be withdrawn from custodial inventory.

- After notified by the DI about the decisions above, the 3PLP takes the following actions (in the given order) by the beginning of Period $t + 1$: (i) fit-sorts the used cash and places the resulting fit cash (resp., unfit cash) in custodial inventory and/or fit-cash inventory (resp., unfit cash inventory) based on the DI's preference, (ii) if required, transports the used cash and/or unfit cash from the DI to the Fed and withdraws fit cash from the Fed. By the beginning of the Period $t + 1$, the DI receives the fit cash (if any) withdrawn from the Fed.

- During Period $t+1$, the DI uses the fit-cash inventory as well as custodial inventory to satisfy demand (from customers) and collects the used cash deposited (by customers).

Note: In RM, the DI maintains two types of fit cash inventories: the regular fit cash in its own vault and the fit cash in custodial inventory (managed by the 3PLP). We use I_t^f (to retain the same notation as in BM) to denote the former and I_t^c to denote the latter.

Notation

For consistency, the basic notation is the same as that defined for BM in Section 3.2.1. We, therefore, define only the additional notation required for RM.

Parameters:

s_r	Setup cost for fit-sorting.
s_c	Setup cost for withdrawal from custodial inventory.
h_c	Per bundle per period custodial inventory holding cost.
r	Per bundle fit-sorting cost.

Decision Variables:

x_t^n	The quantity of unfit cash (in bundles) deposited to the Fed at the end of Period t, $t = 0, 1, \ldots, T$.
x_t^r	The quantity of used cash (in bundles) fit-sorted at the end of Period t, $t = 0, 1, \ldots, T$.
x_t^{c+}	The quantity of fit cash (in bundles) deposited to custodial inventory at the end of Period t, $t = 0, 1, \ldots, T$.
x_t^{c-}	The quantity of fit cash (in bundles) from custodial inventory needed to satisfy fit-cash demand during Period $t+1$, $t = 0, 1, \ldots, T-1$. This decision is made at the end of Period t. For convenience of notation, we let $x_T^{c-} = 0$.
X	The solution vector $(x_t^u, x_t^f, x_t^n, x_t^r, x_t^{c+}, x_t^{c-}), t = 0, 1, \ldots, T$, which includes all the decisions made during the planning horizon.

Other Notation:

y_t^o	$y_t^o = 0$, if $x_t^u = x_t^f = x_t^n = 0$, $t = 0, 1, \ldots, T$; $y_t^o = 1$, otherwise.
y_t^c	$y_t^c = 1$, if $x_t^{c-} > 0$, $t = 0, 1, \ldots, T$; $y_t^c = 0$, otherwise.
y_t^r	$y_t^r = 1$, if $x_t^r > 0$, $t = 0, 1, \ldots, T$; $y_t^r = 0$, otherwise.
I_t^n	The inventory of unfit cash (in bundles) at the end of Period t, $t = 0, 1, \ldots, T$.
I_t^c	The custodial inventory (in bundles) at the end of Period t, $t = 0, 1, \ldots, T$.
H_t^c	The cost charged by the 3PLP for holding custodial inventory during Period t.
O_t^c	The ordering cost charged by the 3PLP for withdrawal from custodial inventory during Period t.

F_t	The cost charged by the 3PLP for fit-sorting cash at the end of Period $t-1$. Without loss of generality, this cost is charged to Period t.
C_t	The total cost incurred during Period t. Thus, $C_t = H_t + H_t^c + O_t + O_t^c + R_t + F_t$.

Cost Calculation

The total cost over the planning horizon is the sum of the following six costs:

1. Holding cost for used cash, (regular) fit cash, and unfit cash: $H = \sum_{t=0}^{T}[0.5h(2I_t^u - d_t^u) + 0.5h(2I_t^f + d_t^f - x_{t-1}^{c-}) + hI_t^n]$.

   Note: Since the inventory of used cash at the end of Period t is I_t^u, and the deposit of used cash during Period t is d_t^u, retrospectively, at the beginning of Period t, the inventory is $I_t^u - d_t^u$. Thus, the average inventory of used cash during Period t is $0.5(2I_t^u - d_t^u)$. Similarly, there is I_t^f regular fit cash at the end of Period t and the customers withdraw d_t^f during Period t. Since the withdrawal of x_{t-1}^{c-} from custodial inventory is also used to satisfy the customers' demand, retrospectively, at the beginning of Period t, the fit-cash inventory is $I_t^f + d_t^f - x_{t-1}^{c-}$. Thus, the average inventory of fit cash during Period t is $0.5(2I_t^f + d_t^f - x_{t-1}^{c-})$. Since there is no change to unfit-cash inventory during a period, the average inventory of used cash during Period t is I_t^n.

2. Custodial inventory holding cost: $H^c = \sum_{t=0}^{T} 0.5h_c(2I_t^c + x_{t-1}^{c-})$.

   Note: Since the custodial inventory at the end of Period t is I_t^c, and x_{t-1}^{c-} custodial inventory is withdrawn during Period t to satisfy the customers' demand, retrospectively, at the beginning of Period t, the custodial inventory is $I_t^c + x_{t-1}^{c-}$. Thus, the average custodial inventory during Period t is $0.5(2I_t^c + x_{t-1}^{c-})$.

3. Ordering cost: $O = \sum_{t=0}^{T} s y_t^o$.

4. Custodial inventory setup cost: $O^c = \sum_{t=0}^{T} s_c y_t^c$.

5. Cross-shipping cost: $R = \sum_{t=0}^{T} e \min\{\rho x_t^u, x_t^f\}$.

6. Fit-sorting cost: $F = \sum_{t=0}^{T} (s_r y_t^r + r x_t^r)$.

3.3.2 Computational Complexity and a Structural Property of RM

From the precise description above, it is intuitive that the decisions in RM are significantly more complex than those in BM. The following result corroborates this intuition by formally establishing the hardness of RM.

Theorem 3.3.1 *The decision problem corresponding to RM is NP-Complete.*

Proof: The NP-Complete problem which we use in our reduction is the PARTITION PROBLEM (Garey and Johnson, 1979).

PARTITION PROBLEM (PP)

INSTANCE: A set of positive integers $A = \{a_1, a_2, \ldots, a_n\}$, a positive integer B, such that $\sum_{i=1}^{n} a_i = 2B$.

SOLUTION: A subset $A_1 \subset A$, such that $\sum_{a_i \in A_1} a_i = B$.

Given an arbitrary instance of PP, we construct an instance of RM as follows. Let $T = n+2$. $d_1^u = \frac{4}{3}B, d_2^u = d_3^u = \ldots = d_{n+2}^u = 0$. $d_1^f = d_2^f = 0, d_3^f = a_1, d_4^f = a_2, \ldots, d_{n+2}^f = a_n$. Let $s_r = 0, r = 0, h_c = 0, s = s_c = 1, h \geq 2n+3$. For the constructed instance, we consider the following decision question.

<u>Decision Question</u>: Let $D = \frac{7}{6}hB + n + 1$. Does there exist a solution X for RM such that $C(X) \leq D$?

The construction of the decision problem from the given instance of PP is polynomially bounded. The decision question, therefore, is in class NP. We now show that the decision question has an affirmative answer if and only if the set A has a subset A_1 such that $\sum_{a_i \in A_1} a_i = B$.

$\Longleftarrow$ Let $A_1 \subset A$ be such that $\sum_{a_i \in A_1} a_i = B$. We let $n_1 = |A_1|$, the number of elements in set A_1. Consider the following solution of the constructed instance of RM: $X = \{x_1^r = \frac{4}{3}B; x_1^{c+} = B; x_1^n = \frac{1}{3}B; x_{1+i}^f = a_i, a_i \in A_1; x_{1+i}^{c-} = a_i, a_i \in A \backslash A_1$; all other decision variables are $0\}$. The solution X can be interpreted as follows. At the end of Period 1, the DI fit-sorts all the used cash it collects during that period, places all the resulting fit cash in custodial inventory, and deposits the unfit cash to the Fed. For $t \in \{3, 4, \ldots, n+2\}$, the DI withdraws just enough fit cash to satisfy the demand of that period either entirely from the Fed at the end of Period $t-1$ or entirely from custodial inventory during Period t.

For the solution X, the holding cost is incurred from two sources: the holding cost for used cash during Period 1 and the holding cost for fit cash during the periods when we use withdrawal from the Fed to satisfy fit-cash demand. Thus, $H(X) = \frac{1}{2}h(d_1^u + \sum_{t=2}^{n+2} x_t^f) = \frac{1}{2}h(\frac{4}{3}B + B) = \frac{7}{6}hB$. Since $h_c = 0$, we have $H^c(X) = 0$. Also, $R(X) = 0, F(X) = 0, O(X) = s(1 + n_1) = 1 + n_1, O^c(X) = s_c(n - n_1) = (n - n_1)$. Thus, the total cost of X is:

$$
\begin{aligned}
C(X) &= H(X) + H^c(X) + R(X) + F(X) + O(X) + O^c(X) \\
&= \frac{7}{6}hB + 0 + 0 + 0 + (1 + n_1) + (n - n_1) = \frac{7}{6}hB + n + 1 = D
\end{aligned}
$$

$\Longrightarrow$ Suppose X is a solution of RM such that $C(X) \leq D$. The following claims help us to characterize the required set A_1.

Claim 3.3.1 $x_t^u = 0, t = 0, 1, \ldots, n+2$.

Proof: Since the sum of all used cash deposits over the planning horizon is an obvious upper bound on the total fit-sorted cash, we have $\sum_{t=0}^{n+1} x_t^r \leq \sum_{t=1}^{n+2} d_t^u = \frac{4}{3}B$. The fit cash placed in custodial inventory has to necessarily come from fit-sorting; thus, $\sum_{t=0}^{n+1} x_t^{c-} \leq \sum_{t=0}^{n+1} x_t^{c+} \leq \frac{3}{4}\sum_{t=0}^{n+1} x_t^r = B$. Also since the demand of fit cash has to be satisfied by withdrawals either from the Fed or from custodial inventory, we have $\sum_{t=0}^{n+1} x_t^f + \sum_{t=0}^{n+1} x_t^{c-} = \sum_{t=1}^{n+2} d_t^f = \sum_{i=1}^{n} a_i = 2B$. Consequently, we have $\sum_{t=0}^{n+1} x_t^f = 2B - \sum_{t=0}^{n+1} x_t^{c-} \geq 2B - B = B$. When we withdraw x_t^f fit cash from the Fed at the end of Period t, we incur a holding cost of at least $\frac{1}{2}hx_t^f$ during Period $t+1$. Similarly, we incur a holding cost of at least $\frac{1}{2}hd_1^u$ for used cash during Period 1. Thus,

$$C(X) \geq \frac{1}{2}h\sum_{t=0}^{n+1} x_t^f + \frac{1}{2}hd_1^u = \frac{1}{2}h\sum_{t=0}^{n+1} x_t^f + \frac{2}{3}hB.$$

Assume $\sum_{t=0}^{n+1} x_t^f > B$. Then, we have $\sum_{t=0}^{n+1} x_t^f \geq B+1$ since all decision variables are integral. Since $h \geq 2n+3$, we have

$$C(X) \geq \frac{2}{3}hB + \frac{1}{2}h\sum_{t=0}^{n+1} x_t^f \geq \frac{2}{3}hB + \frac{1}{2}hB + \frac{1}{2}h > \frac{7}{6}hB + n + 1 = D,$$

which contradicts $C(X) \leq D$. Therefore, $\sum_{t=0}^{n+1} x_t^f = B$. Then, $\sum_{t=0}^{n+1} x_t^{c-} = 2B - \sum_{t=0}^{n+1} x_t^f = B$. Consequently, $\sum_{t=0}^{n+1} x_t^{c+} = B$ and $\sum_{t=0}^{n+1} x_t^r = \frac{4}{3}B$. Since $\sum_{t=0}^{n+1} x_t^r + \sum_{t=0}^{n+2} x_t^u = \sum_{t=1}^{n+2} d_t^u$, we have $\sum_{t=0}^{n+2} x_t^u = \frac{4}{3}B - \frac{4}{3}B = 0$. Thus, $x_t^u = 0, t = 0, 1, \ldots, n+2$. □

Claim 3.3.2 $x_0^f = x_1^f = x_0^{c-} = x_1^{c-} = 0$; $x_1^r = \frac{4}{3}B$, $x_t^r = 0, t = 2, 3, \ldots, n+2$; $x_1^{c+} = B$, $x_t^{c+} = 0, t = 2, 3, \ldots, n+2$; $x_1^n = \frac{1}{3}B$, $x_1^n = 0, t = 2, 3, \ldots, n+2$.

Proof: From Claim 3.3.1, we have $C(X) \geq \frac{2}{3}hB + \frac{1}{2}h\sum_{t=0}^{n+1} x_t^f = \frac{7}{6}hB$. Moreover, a holding cost of at least $\frac{1}{2}hI_t^f$ (resp., $\frac{1}{2}hI_t^u$, $\frac{1}{2}hI_t^n$) is incurred on the fit-cash inventory (resp., used-cash inventory, unfit-cash inventory) during Period t. Thus,

$$C(X) \geq \frac{7}{6}hB + \frac{1}{2}h(\sum_{t=1}^{n+2} I_t^f + \sum_{t=1}^{n+2} I_t^n + \sum_{t=2}^{n+2} I_t^u).$$

Using an argument similar to that in the proof of Claim 6, we have $\sum_{t=1}^{n+2} I_t^f + \sum_{t=1}^{n+2} I_t^n + \sum_{t=2}^{n+2} I_t^u = 0$. Thus, $I_t^f = 0, t = 1, 2, \ldots, n+2$; $I_t^n = 0, t = 1, 2, \ldots, n+2$; $I_t^u = 0, t = 2, 3, \ldots, n+2$.

Since $I_1^f = I_2^f = 0$, $d_1^f = d_2^f = 0$, there is no withdrawal of fit cash either from the Fed at the end of Periods 0 and 1 or from custodial inventory during the Periods 1 and 2, i.e., $x_0^f = x_1^f = x_0^{c-} = x_1^{c-} = 0$. Since $0 = I_2^u = I_1^u - x_1^u - x_1^r + d_2^u = \frac{4}{3}B - 0 - x_1^r + 0$, we have $x_1^r = \frac{4}{3}B$. From Claim 3.3.1, we have $\sum_{t=0}^{n+1} x_t^r = \frac{4}{3}B$; thus, $x_t^r = 0, t = 2, 3, \ldots, n+2$. Since $x_t^{c+} \leq x_t^r$, we have $x_t^{c+} = 0, t = 2, 3, \ldots, n+2$. Since $0 = I_2^f = I_1^f + 0.75x_1^r - x_1^{c+} + x_1^f + x_1^{c-} - d_2^f = 0 + B - x_1^{c+} + 0 + 0 - 0$, we have $x_1^{c+} = B$. Using $0 = I_{t+1}^n = I_t^n + 0.25x_t^r - x_t^n$, we have $x_1^n = \frac{1}{3}B$, $x_t^n = 0, t = 2, 3, \ldots, n+2$. $\square$

Claim 3.3.3 $\sum_{t=2}^{n+1} x_t^f = B, \sum_{t=2}^{n+1} x_t^{c-} = B$. *If* $x_t^{c-} > 0$, *then* $x_t^{c-} = d_{t+1}^f$, $2 \leq t \leq n+1$.

Proof: From Claim 3.3.1, we have $\sum_{t=0}^{n+1} x_t^f = B, \sum_{t=0}^{n+1} x_t^{c-} = B$. From Claim 3.3.2, we have $x_0^f = x_1^f = x_0^{c-} = x_1^{c-} = 0$. Thus, $\sum_{t=2}^{n+1} x_t^f = B$ and $\sum_{t=2}^{n+1} x_t^{c-} = B$.

Since $I_t^f = 0, t = 1, 2, \ldots, n+2$, we do not have positive fit cash inventory at the end of these periods. Then, for each period $t \in \{3, 4, \ldots, n+2\}$, since $d_t^f > 0$, fit cash is withdrawn either from the Fed at the end of Period $t-1$ or from custodial inventory during Period t to satisfy the fit-cash demand of that period. Thus, for $t \in \{3, 4, \ldots, n+2\}$, we incur either the ordering cost or setup cost for withdrawal from custodial inventory (or both) during Period t. Thus, $O_t(X) + O_t^c(X) \geq 1$, for $3 \leq t \leq n+2$. Suppose, in X, there exists some $\tilde{t}$ such that $x_{\tilde{t}}^{c-} > 0$ and $x_{\tilde{t}}^{c-} < d_{\tilde{t}+1}^f$. Then, we have $x_{\tilde{t}}^f = d_{\tilde{t}+1}^f - x_{\tilde{t}}^{c-} > 0$. Thus, during Period $\tilde{t}+1$, both the setup costs s and s_c are incurred, i.e., $O_{\tilde{t}}(X) + O_{\tilde{t}}^c(X) = 2$. Thus, $C(X) \geq \frac{7}{6}hB + 1 + 2 + (n-1) > D$, which contradicts $C(X) \leq D$. The result now follows. $\square$

Let $I = \{i | x_{i+1}^{c-} > 0\}$, $A_1 = \{a_i | i \in I\}$. As a consequence of Claim 3.3.3, we have $\sum_{t=2}^{n+1} x_t^{c-} = \sum_{i \in I} d_{i+2}^f = B$. Since $d_{i+2}^f = a_i, i = 1, 2, \ldots, n$, we have $\sum_{a_i \in A_1} a_i = B$. This concludes the proof of Theorem 3.3.1. ∎

We now prove a structural property of RM.

Theorem 3.3.2 *If $r < e\rho$, then there exists an optimal solution, say X, of RM such that $x_t^f = 0$ or $x_t^u = 0$ or $x_t^r = 0$ for $t = 1, 2, \ldots, T$.*

Proof: Consider an arbitrary solution X of RM and let t be the period such that $x_t^f > 0$, $x_t^u > 0$, and $x_t^r > 0$. Let $\gamma = \min\{\frac{1}{\rho}x_t^f, x_t^u\}$ and consider another solution $\bar{X}$ of RM, defined as follows. Let $\bar{x}_t^u = x_t^u - \gamma; \bar{x}_t^r = x_t^r + \gamma; \bar{x}_t^f = x_t^f - \rho\gamma; \bar{x}_t^n = x_t^n + (1-\rho)\gamma; \bar{x}_i^u = x_i^u, \bar{x}_i^r = x_i^r, \bar{x}_i^f = x_i^f, \bar{x}_i^n = x_i^n, i \in \{1, 2, \ldots, T\} \backslash \{t\}; \bar{x}_i^{c+} = x_i^{c+}; \bar{x}_i^{c-} = x_i^{c-}, i = 1, 2, \ldots, T$. Thus, we have either $\bar{x}_t^u = 0$ or $\bar{x}_t^f = 0$. Since $x_t^u > 0$ and $\bar{x}_t^n > 0$, we have $O_{t+1}(\bar{X}) - O_{t+1}(X) = s - s = 0$. Also, $x_t^r > 0$ and $\bar{x}_t^r = x_t^r + \gamma > 0$. Hence, we have $F_{t+1}(\bar{X}) - F_{t+1}(X) = (s_r + r\bar{x}_t^r) - (s_r + rx_t^r) = r\gamma$. Next, note that $R_{t+1}(\bar{X}) - R_{t+1}(X) = e(\min\{\rho x_t^u - \rho\gamma, x_t^f - \rho\gamma\} - \min\{\rho x_t^u, x_t^f\}) = -e\rho\gamma$. Furthermore, $H(\bar{X}) - H(X) = 0$, $H^c(\bar{X}) - H^c(X) = 0$, and $O^c(\bar{X}) - O^c(X) = 0$. Therefore, $C(\bar{X}) - C(X) = O_{t+1}(\bar{X}) - O_{t+1}(X) + F_{t+1}(\bar{X}) - F_{t+1}(X) + R_{t+1}(\bar{X}) - R_{t+1}(X) = (r - e\rho)\gamma$. Hence, if $r < e\rho$, we have $C(\bar{X}) - C(X) < 0$, which implies $\bar{X}$ is a better solution than X.

Thus, in the solution X of RM, if $r < e\rho$ and there exists some t with $x_t^f > 0$, $x_t^u > 0$, and $x_t^r > 0$, then it is possible to obtain a better solution $(\bar{X})$ in which at least one of x_t^f, x_t^u, and x_t^r, is zero for $t = 1, 2, \ldots, T$. To conclude the proof, note that the above argument can be applied repeatedly if there exist multiple periods that satisfy $x_t^f > 0$, $x_t^u > 0$, and $x_t^r > 0$. ■

Theorem 3.3.2 can be intuitively interpreted in the following manner: If the cost of fit-sorting one extra bundle of used-cash (i.e., r, the per bundle fit-sorting cost) is less than the marginal saving from using the resulting fit cash to reduce cross-shipping (i.e., $e\rho$, the product of the per bundle cross-shipping cost and the percentage of fit cash in the used cash), then a DI will not both deposit used-cash and withdraw fit-cash in the same period. For otherwise, fit-sorting an extra bundle and, thereby, reducing the amount of fit-cash withdrawal results in a better solution. Using this argument iteratively, we can conclude that either the quantity of used-cash deposit or the quantity of fit-cash withdrawal becomes 0, which implies that cross-shipping is eliminated in that period.

Theorem 3.3.1 above, together with the significant recent advances in computational integer programming motivate us to formulate RM as a mixed-integer program.

3.3.3 Formulating RM as a Mixed-Integer Program (MIP)

We retain most of the notation from Sections 3.2.1 and 3.3.1, and only define the additional notation required for the formulation.

Additional Notation:

z_t	$z_t = 1$, if $x_t^f \leq x_t^u, t = 0, 1, \ldots, T$; $z_t = 0$, otherwise.

The Objective Function

Our objective is to minimize the sum of six types of costs (see Section 3.3.1), i.e.,

$$\min\ (O + O^c + H + H^c + F + R)$$

Constraints

We have five main classes of constraints.

1. **Inventory Balance Constraints**:

   Recall that all the inventories are counted at the end of each period and before any decisions are taken. There are four types of inventories.

   (i) Used Cash Inventory Balance:

   $$I_{t+1}^u = I_t^u - x_t^r - x_t^u + d_{t+1}^u \qquad t = 0, 1, \ldots, T-1$$

   (ii) Unfit Cash Inventory Balance:

   $$I_{t+1}^n = I_t^n + 0.25x_t^r - x_t^n \qquad t = 0, 1, \ldots, T-1$$

   (iii) Fit Cash Inventory Balance:

   $$I_{t+1}^f = I_t^f + 0.75x_t^r - x_t^{c+} + x_t^f + x_t^{c-} - d_{t+1}^f \qquad t = 0, 1, \ldots, T-1$$

   (iv) Custodial Inventory Balance:

   $$I_{t+1}^c = I_t^c + x_t^{c+} - x_t^{c-} \qquad t = 0, 1, \ldots, T-1$$

2. **Cross-Shipping Constraints**:

   Recall that $R_{t+1} = e \min\{0.75x_t^u, x_t^f\}$. Since we are minimizing the objective function, this constraint can be replaced by the following two constraints:

$$R_{t+1} \geq 0.75e(x_t^u - z_t \sum_{t=1}^{T} d_t^u) \qquad t = 0, 1, \ldots, T-1$$

$$R_{t+1} \geq e[x_t^f - (1 - z_t) \sum_{t=1}^{T} d_t^f] \qquad t = 0, 1, \ldots, T-1$$

3. **Constraints for Fit-Sorting**:

   At the end of each period, the sum of the quantity fit-sorted and the quantity of used-cash deposited cannot exceed the available used-cash inventory.

$$x_t^u + x_t^r \leq I_t^u \qquad t = 0, 1, \ldots, T$$

   The quantity of fit cash placed in custodial inventory cannot exceed the quantity of fit-sorted cash.

$$x_t^{c+} \leq 0.75x_t^r \qquad t = 0, 1, \ldots, T-1$$

4. **Constraints for Computing Setup and Ordering Costs**:

   We have 3 types of setup and ordering costs:

   (i) At the end of Period t, an ordering cost is charged if the DI requests the 3PLP to conduct any of the following three activities: (1) deposits used cash to the Fed, (2) deposits unfit cash to the Fed, (3) withdraws fit cash from the Fed.

   Thus,

$$x_t^u + x_t^n \le y_t^o \sum_{t=1}^{T} d_t^u \qquad t = 0, 1, \ldots, T-1$$

$$x_t^f \le y_t^o \sum_{t=1}^{T} d_t^f \qquad t = 0, 1, \ldots, T-1$$

(ii) At the end of Period t, a setup cost for custodial inventory is charged if the DI requests to withdraw from its custodial inventory. Thus,

$$x_t^{c-} \le 0.75 y_t^c \sum_{t=1}^{T} d_t^u \qquad t = 0, 1, \ldots, T-1$$

(iii) At the end of Period t, a setup cost for fit-sorting is charged if the DI requests the 3PLP to conduct fit-sorting. Thus,

$$x_t^r \le y_t^r \sum_{t=1}^{T} d_t^u \qquad t = 0, 1, \ldots, T-1$$

5. **Non-Negativity and 0-1 Restrictions**:

$$\begin{aligned} x_t^u, x_t^f, x_t^r, x_t^{c+}, x_t^n, x_t^{c-} &\ge 0 \qquad t = 0, 1, \ldots, T-1 \\ I_t^f, I_t^c, I_t^n &\ge 0 \qquad t = 0, 1, \ldots, T-1 \\ y_t^o, y_t^c, y_t^r, z_t &\in \{0, 1\} \qquad t = 0, 1, \ldots, T-1 \end{aligned}$$

As will be seen in the next section, CPLEX (version 11.1.1) is able to solve realistic instances of RM in reasonable time. Our ability to solve instances of BM and RM to optimality allows us to evaluate the impact of local reuse of cash and derive a variety of insights on the forces that influence the volume of cross-shipping. This will be our goal in the next section.

3.4 Understanding the Value of Local Reuse and Influences on the Volume of Cross-Shipping

Recall that a DI may choose not to fit-sort its used-cash deposits and, instead, opt to pay recirculation fees for cross-shipping. To resolve the question of whether or not to fit-sort, it becomes important for managers of DIs to quantify the impact of fit-sorting and participating in the custodial inventory program. Subsequently, if the DI decides to fit-sort, the optimal amount to fit-sort and the influence of the problem parameters on the volume of cross-shipping helps understand the various tradeoffs involved. We address these issues via a comprehensive numerical study.

An ideal metric, which we refer to as the **Value of Local Reuse** (VoR), to capture the monetary impact from implementing fit-sorting and custodial inventory is the percentage cost saving offered by RM over BM. That is,

$$\text{VoR} = \frac{(\text{Objective Value of BM - Objective Value of RM}) \times 100\%}{\text{Objective Value of BM}}.$$

Similarly, the **Extent of Local Reuse** (EoR) is the percentage of fit cash demand satisfied by fit-sorting (i.e., by local recirculation) in RM. Since about 75% of the used-cash is fit-cash (Section 3.1), we have

$$\text{EoR} = \frac{(\text{Total Quantity Fit-Sorted over the Planning Horizon}) \times 75\%}{\text{The Total Demand of Fit Cash over the Planning Horizon}}.$$

Another characteristic that is expected to influence the extent of cross-shipping for a DI is the imbalance between its average per-period deposit and demand. Accordingly, we define a DI's **Imbalance Ratio**, α, as follows:

$$\alpha = \frac{\text{Average Per-Period Used Cash Deposit}}{\text{Average Per-Period Fit-Cash Demand}}.$$

We first describe our test bed in Section 3.4.1 and then discuss the results and insights in Section 3.4.2.

3.4.1 The Test Bed

To ground our test bed in reality, we use information from the Fed's publicly-available documents (Federal Reserve 2006, 2009) and also from the relevant literature (Mehrotra et al. 2010, Dawande et al. 2010).

As mentioned in Section 3.1, the weekly demand of fit cash for medium-size DIs is typically between 1000 to 2000 bundles. Thus, we choose the weekly demand of fit cash (in bundles) randomly from $U[1000, 2000]$. As observed in Mehrotra et al. (2010), discussions with practitioners indicate that there is little week-to-week variation in the value of the ratio α (defined above) for a specific DI; however, across DIs, the value of α may vary and typically ranges between 1 and 3. Accordingly, we consider four values of α: 1, 1.5, 2, and 3. Thus, the weekly deposit of used cash is generated randomly from $U[1000\alpha, 2000\alpha]$. We consider three possible values for the length of the planning horizon, T: 5, 10, 15.

To evaluate the impact of the cross-shipping fee e on the optimal solution, we consider three values: \$5 (the current value), \$6, and \$7. The dominant component of a DI's per-bundle holding cost, h, of fit cash is the cost of lost opportunity of lending via the Federal Funds Market. Between 1994 to 2008, the annual Federal Funds Rate has varied between a minimum of 1.13% and a maximum of 6.24% (Federal Reserve 2010a). Currently (2009 and early 2010), this rate is below 1%, influenced largely by the economic downturn that started in late 2007. Assuming a rate of 1%, the holding cost per-bundle per-period (week) is $\$20 \times 1000 \times 1\%/50 = \$4$. Anticipating a moderate rise of this rate in the future, we consider two additional values: 3% and 5%. Thus, we consider three values of h: \$4, \$12, and \$20. Based on discussions with a secure-logistics provider, we choose three values of the fixed ordering cost, s, of transportation: \$1500, \$2000, \$2500. For a fair comparison of the two models (BM and RM) for assessing the value and extent of reuse, we avoid varying those parameters that are relevant only to RM, namely h_c, r, s_r, and s_c (Section 3.3.1). Recall that deposits to custodial inventory are credited to the DI's account with the Fed and can be lent (to other institutions) by the DI. Hence, such deposits incur no cost of lost lending opportunity. Consequently, the per-bundle per-period holding cost of custodial inventory, h_c, is only the physical storage cost, which we set at \$1. The per-bundle fit-sorting fee, r, is set at \$2. Both the setup costs s_c and s_r are administrative fees that are expected to be modest; we fix them to \$200 and \$1000, respectively. For each combination of the horizon-length T and the imbalance ratio α, we generate 5 instances of demand data. For each instance of demand data, we have $3 \times 3 \times 3 = 27$ combinations of the parameter settings above, for a total of $3 \times 4 \times 5 \times 27 = 1620$ instances.

In Section 3.4.2, when assessing the sensitivity of the cross-shipping volume, we fix e at \$5 and the values of T and α at their baseline values of 10 and 1.5, respectively, and vary only the values of s, h, and r. Thus, the total number of instances considered for this second experiment is $5 \times 3 \times 3 \times 4 = 180$.

Table 3.1. Parameter Settings to Quantify the Value of Local Reuse and the Sensitivity of the Volume of Cross-Shipping.

Parameter	*Values*	*Parameter*	*Values*	*Parameter*	*Value*
T	5, 10, 15	h	\$4, \$12, \$20	s_r	\$1000
α	1, 1.5, 2, 3	s	\$1500, \$2000, \$2500	h_c	\$1
r	\$2, \$3, \$4, \$5	e	\$5, \$6, \$7	s_c	\$200

Table 3.1 summarizes the representative values of the parameters used in our computational study.

3.4.2 Results

CPLEX (version 11.1.1) was able to optimally solve the MIPs corresponding to the instances of RM in a reasonable amount of time. The instances of BM were solved using the dynamic programming formulation of Section 3.2.

The Value and Extent of Local Reuse

For the instances in our test bed, VoR ranges between 14.9% and 36.4%. Over all the 1620 instances, both the mean and median are around 26.6%. EoR ranges between 41.3% and 100%, with mean and median values of 82.2% and 90.5%, respectively. We, therefore, conclude that *the saving for a DI from implementing fit-sorting and custodial inventory as well as the extent of local recirculation are substantial.* Indeed, our results confirm the ability of the Fed's policy to generate a welfare-like situation. From the viewpoint of DIs, substantial reuse is beneficial under the Fed's policy since this is the optimal choice. In turn, the 3PLPs benefit from the new fit-sorting business opportunity. Finally, the Fed too furthers its goal of reducing the social cost of providing currency to the public.

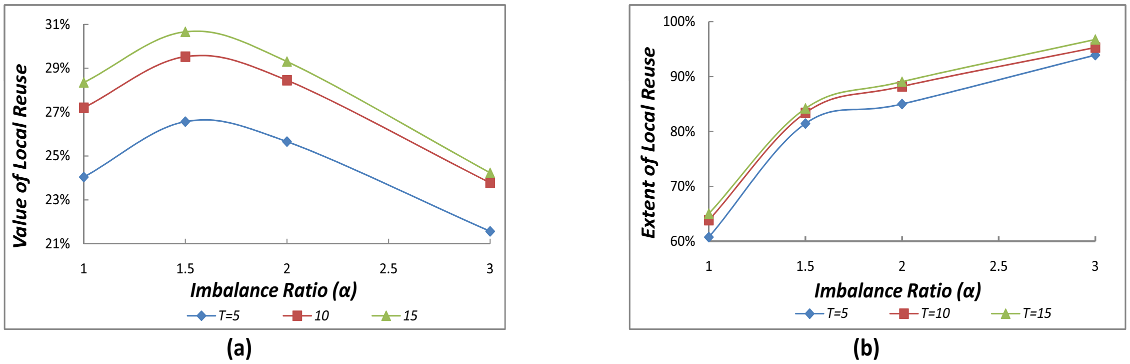


Figure 3.2. Impact of Imbalance and Length of Planning Horizon on Value and Extent of Local Reuse.

The imbalance ratio, α, is a good metric to highlight the relative differences between the behavior of VoR and EoR. We have VoR at its highest when $\alpha = 1.5$ (Figure 3.2(a)). To understand this, consider a scenario where the used-cash deposit in a week is consistently about $1/75\% = 1.33$ times the fit-cash demand in the next week. Then, by fit-sorting all the used-cash deposit each week, the DI can satisfy its fit-cash demand and completely eliminate the transportation – to and from the Fed – of fit cash and used cash. This simple observation is quite helpful. When the per-week supply of used cash is about the same as the demand for fit cash (i.e., $\alpha = 1$), the DI cannot generate enough fit cash from local reuse to meet its demand. Recall that about 75% of the quantity fit-sorted is fit-cash. Thus, for $\alpha = 1$, EoR has an upper bound of around 75% (Figure 3.2(b)), which implies that a maximum of 75% of the fit-cash demand is satisfied by fit-sorting. Thus, it becomes necessary for the DI to withdraw fit cash from the Fed and, consequently, incur transportation cost. On the other hand when α is too high (say, $\alpha \in \{2, 3\}$), although the DI can satisfy all of its fit-cash demand by local reuse (i.e., EoR ≈ 1), there is too much used cash at hand. This excess used cash must be deposited to the Fed which, again, implies an increase in transportation cost. Thus, the DI incurs additional costs at extreme values of α. When $\alpha = 1.5$, the deposit of used cash is just of the correct magnitude to enable the DI to generate the required fit cash by local reuse. Thus, in this latter case, the shipment activity (if any) between the DI and the Fed are much less frequent than that in the former extreme cases. To summarize, *VoR is at its highest when the imbalance between the used-cash deposit and fit-cash demand is just enough for the DI to effectively satisfy all demand via local reuse.*

Another observation from Figure 3.2(a) is that *VoR increases with the length of planning horizon*. This is intuitive, since a longer planning horizon implies more opportunities to combine orders and enable better decisions. Figure 3.2(b) also shows that even with enough used cash (i.e., $\alpha > 1.5$), the DI still does not fully use local reuse to satisfy its fit-cash demand (EoR ≈ 0.80). The reason is that a DI may prefer to fit-sort less when the per-bundle per-period holding cost of fit cash (h) is low. Fit-sorting enables the DI to store fit cash in custodial inventory, which has a much lower holding cost than the regular inventory. However, to exploit this, the DI may have to fit-sort in advance and keep fit cash in the custodial inventory for multiple periods. On the other hand, fit cash ordered from the Fed needs to be kept in the regular inventory but for a shorter time (e.g., one period). When h is low, the cost of holding fit cash for a longer time in custodial inventory can exceed that of holding it for a shorter time in regular inventory. Thus, withdrawing fit cash from the Fed to satisfy fit-cash demand becomes preferable.

Influences on the Volume of Cross-Shipping

To understand the combination of forces that influence the volume of cross-shipping, we first examine the impact of the fit-sorting cost on the optimal solution. When the per-bundle cost of fit-sorting increases, if the DI intends to maintain the same level of reuse, then it is immediate that the total fit-sorting cost will increase. Thus, one option to reduce the total cost is to reduce the fit-sorting volume over the planning horizon. This reduces one or both of the fixed and variable costs of fit-sorting. However, reducing the fit-sorting volume implies the following two consequences: (1) with less fit cash from fit-sorting, the DI has to withdraw more fit cash from the Fed to meet its demand and (2) the DI has more inventory of used cash which needs to be eventually deposited to the Fed. In turn, these outcomes can increase the total cost in two possible ways: (i) if the DI decides to deposit used cash and withdraw fit cash near-simultaneously (i.e., within the same week), then the cross-shipping cost and (possibly) the ordering cost will increase, (ii) if the DI can avoid cross-shipping by either postponing the deposit of used cash or withdrawing fit cash in advance, the holding cost will increase. Thus, the DI faces a healthy tradeoff from reducing the fit-sorting volume.

The previous discussion suggests that the combined influence of per-bundle fit-sorting cost (r) and the per-bundle holding cost (h) may be the force that decides the volume of cross-shipping. Comparing the optimal solutions of RM over different values of r and h leads to two interesting observations: (1) when the per-bundle holding cost is very low, the volume of cross-shipping, the total holding cost as well as the number of orders increase with fit-sorting cost, while the volume of fit-sorting and the volume in custodial inventory decrease with fit-sorting cost, (2) when the per-bundle holding cost is very high, RM retains the same optimal solution even when the fit-sorting cost varies. We now provide an intuitive explanation for these two observations: When r has its minimum value, we have $r \ll \min\{h, e\}$. Thus, fully exploiting fit-sorting to satisfy fit-cash demand is most economical for the DI. When

r increases, the DI may consider the option of reducing the fit-sorting volume, which, as discussed above, will increase either the holding cost or the cross-shipping cost. When the per-bundle holding cost (h) is very high, it still significantly dominates the increased r. Thus, increasing inventory cost becomes uneconomical. The other option – increasing cross-shipping – leads to an increase that dominates the savings resulting from the reduction in the fixed and variable costs of fit-sorting. Consequently, the DI chooses to maintain the maximum level of fit-sorting, resulting in the same optimal solution. However, when the per-bundle holding cost is low, the DI can tolerate an increase in inventory costs. Therefore, as r increases, the progressively larger impact of reducing the fit-sorting cost (by reducing the fit-sorting volume) begins to dominate the combined increase in the ordering, holding, and cross-shipping costs. As a result, the DI chooses to fit-sort less and, instead, increase the volume of cross-shipping. To summarize, *the impact of changing the fit-sorting cost on the volume of cross-shipping is governed by the ratio of the holding cost and the fit-sorting cost. As long as the holding cost continues to significantly dominate fit-sorting cost, there is little or no change in the volume of cross-shipping as the fit-sorting cost increases. When the holding cost is comparable to fit-sorting cost, an increase in the fit-sorting cost leads to a consistent increase in the volume of cross-shipping.*

Figure 3.3 confirms our interpretation. The percentage of instances with zero cross-shipping volume for each value of r and h is illustrated in Figure 3.3(a). At a high holding cost, we have a complete absence of cross-shipping as the per-bundle fit-sorting cost varies over its range. However, when the holding cost is low, the percentage of instances with zero cross-shipping cost decreases with the fit-sorting cost. To further explore the latter case, we illustrate the

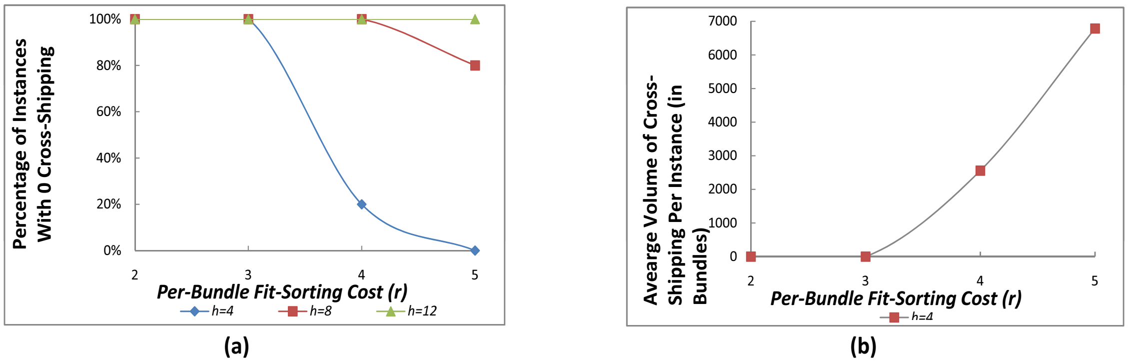


Figure 3.3. Impact of Per-Bundle Fit-Sorting Cost (r) and Per-Bundle Holding Cost (h) on Cross-Shipping.

average volume of cross-shipping when the holding cost is low in Figure 3.3(b). As can be clearly seen, the marginal increase in the average volume of cross-shipping increases with an increase in the fit-sorting cost.

Our discussion thus far assumes that the weekly values of used-cash deposit and fit-cash demand during the planning horizon can be forecasted with reasonable accuracy. To adapt the decisions from our models to a real-time scenario, we now develop a rolling-horizon procedure that allows for a limited extent of uncertainty in these two forecasts.

3.5 A Rolling-Horizon Procedure for Real-Time Decisions

In this section, we assume that the actual value of the deposit (resp., demand) in a week varies between a pair of lower and upper bounds (that depend on the week). These bounds can be forecasted at the beginning of the planning horizon and updated throughout the horizon (if necessary). Also, in practice, managers prefer a scheme which can incorporate information (e.g., realized demand) as and when it becomes available. Thus, a rolling-horizon procedure which offers an iterative decision-making environment is an ideal technique to adapt our models for real-time decisions. Under this procedure, we first solve a τ-period problem, for some constant $\tau \geq 1$, by using the initial inventories and the forecasted demand/deposit data for Periods 1 through τ. Although the decisions for the first τ periods are then available, we only implement the decisions for the first period. After the actual demand/deposit for the first period are realized, we observe the inventories at the end of Period 1 and solve another τ-period problem by using the (possibly updated) demand/deposit data for Periods 2 through $\tau + 1$. The iterative process continues in this manner and can be applied over an infinite horizon or a finite, T-period horizon. The τ-period problem encountered in each iteration is referred to as the rolling-horizon problem.

In this section, we first describe the construction of the rolling-horizon problem (Section 3.5.1) and then provide an iterative procedure that adapts solutions of this problem for real-time decisions (Section 3.5.2). Finally, we show – if the widths of the interval forecasts over the planning horizon are modest – that our procedure provides near-optimal, real-time solutions to arbitrary realized instances (Section 3.5.3).

3.5.1 A τ-Period Rolling-Horizon Problem Based on Forecasted Data

We retain the same basic notation as in Section 3.3.1 and only define the additional notation needed.

For the first period of a τ-period rolling horizon problem, if the realized fit-cash demand exceeds the projected demand, decisions that are based on the projected demand but are used for the realized demand may cause an infeasibility (due to lack of the required fit cash). One

Additional Notation:

$\kappa_t^u, \bar{d}_t^u, \nu_t^u$	The forecasted lower bound, average, and upper bound of deposit of used cash (in bundles) from the DI's customers during Period t, $t = 1, 2, \ldots, T$.
$\kappa_t^f, \bar{d}_t^f, \nu_t^f$	The forecasted lower bound, average, and upper bound of demand of fit cash (in bundles) for the DI's customers during Period t, $t = 1, 2, \ldots, T$.
η_t^u	The actual deposit of used cash (in bundles) from the DI's customers during Period t, $t = 1, 2, \ldots, T$.
η_t^f	The actual demand of fit cash (in bundles) for the DI's customers during Period t, $t = 1, 2, \ldots, T$.

simple way to avoid such an infeasibility is to develop a solution that uses the upper bound of fit-cash demand for the first period. Since we need to implement only the decisions for the first period, we can safely use the forecasted average fit-cash demands for the remaining $\tau - 1$ periods. Also, we use the forecasted average used-cash deposits for the τ periods. Thus, given forecasted used-cash deposits of the first τ periods, $(\kappa_t^u, \bar{d}_t^u, \nu_t^u), 1 \leq t \leq \tau$, and forecasted fit-cash demands of the first τ periods, $(\kappa_t^f, \bar{d}_t^f, \nu_t^f), 1 \leq t \leq \tau$, we generate a τ-period rolling-horizon problem using the following data: $d_t^u = \bar{d}_t^u$, $1 \leq t \leq \tau$; $d_1^f = \nu_1^f$; $d_t^f = \bar{d}_t^f$, $2 \leq t \leq \tau$.

For BM, given the initial inventories I_0^u and I_0^f of used and fit cash, respectively, and the deposit/demand data described above, we use the forward recursion described in Section 3.2.4 to solve the τ-period rolling-horizon problem. Similarly, for RM, given the initial inventories of used cash, fit cash, unfit cash, and custodial inventory (i.e., $I_0^u, I_0^f, I_0^n, I_0^c$, respectively), and the deposit/demand data, we use the Mixed-Integer Program described in Section 3.2.4 to solve the rolling-horizon problem.

3.5.2 A Rolling-Horizon Procedure

We now formally describe the rolling-horizon procedure for BM. The procedure for RM is similar, except that we need to calculate two more types of inventories (I_t^n and I_t^c) and record four more types of decision variables (x_t^r, x_t^{c+}, x_t^n, and x_t^{c-}); see Section 3.3.1. Therefore, for brevity, we avoid an explicit description of the procedure for RM.

Procedure ROLLING HORIZON

Step 1. Let $i = 1$.

Step 2. Forecast the used-cash deposits and fit-cash demands for the next τ periods, i.e., $(\kappa_t^u, \bar{d}_t^u, \nu_t^u)$, $(\kappa_t^f, \bar{d}_t^f, \nu_t^f), i \leq t \leq i + \tau - 1$.

Step 3. Generate a τ-period problem, P_b, using $\hat{I}_0^u = I_{i-1}^u, \hat{I}_0^f = I_{i-1}^f$; $\hat{d}_j^u = \bar{d}_{i+j-1}^u, 1 \leq j \leq \tau$; $\hat{d}_1^f = \nu_i^f$; $\hat{d}_j^f = \bar{d}_{i+j-1}^f$, $2 \leq j \leq \tau$.

Step 4. Solve P_b, implement the decisions for the first period, i.e., $\hat{x}_0^u, \hat{x}_0^f$, and record $x_{i-1}^u = \hat{x}_0^u$, $x_{i-1}^f = \hat{x}_0^f$.

Step 5. The actual used-cash deposit of Period i, η_i^u, and the actual fit-cash demand of Period i, η_i^f are realized. Record η_i^u and η_i^f. Observe the ending inventories of used-cash and fit-cash for Period i, $I_i^u = I_{i-1}^u - x_{i-1}^u + \eta_i^u$, $I_i^f = I_{i-1}^f + x_{i-1}^f - \eta_i^f$.

Step 6. If $i = T$, then exit; otherwise, $i = i + 1$, go to Step 2.

At the end of the Period T, we have the realized deposit/demand for the entire T-period planning horizon, i.e., $\eta_i^u, \eta_i^f, 1 \leq i \leq T$. For comparison, we define P_a as the instance of BM corresponding to the realized deposit/demand data (i.e., under the assumption that all realized data is known with certainty a priori). By construction, the solution $(x_i^u, x_i^f), 0 \leq i \leq T - 1$ generated by Procedure ROLLING HORIZON is a feasible solution to Problem P_a.

3.5.3 The Performance of Procedure ROLLING HORIZON

Except for the length of the planning horizon and the demands/deposits, we set the other parameters of BM/RM to their baseline values of the test bed described in Section 3.4.1. We set T to 100 as an approximation of a long-term horizon. For $t = 1, 2, \ldots, T$, the average demands, $\bar{d}_t^f$, of fit cash (in bundles) during Period t are randomly generated as before from $U[1000, 2000]$. Corresponding to $\alpha = 1.5$, the average deposits, $\bar{d}_t^u$, of used cash (in bundles) are randomly generated from $U[1500, 3000]$. To indicate the varying accuracies of the lower/upper bounds of the intervals in which the actual demands/deposits lie, we consider symmetric intervals with widths (w; see Table 3.2) of 5%, 10% and 15%, around the average demands/deposits. That is, we consider three intervals for the actual fit-cash demand η_t^f: $[0.975\bar{d}_t^f, 1.025\bar{d}_t^f]$, $[0.95\bar{d}_t^f, 1.05\bar{d}_t^f]$ and $[0.925\bar{d}_t^f, 1.075\bar{d}_t^f]$. Similarly, we consider three intervals for the actual used-cash deposit η_t^u: $[0.975\bar{d}_t^u, 1.025\bar{d}_t^u]$, $[0.95\bar{d}_t^u, 1.05\bar{d}_t^u]$ and $[0.925\bar{d}_t^u, 1.075\bar{d}_t^u]$. We consider three values of the length, τ, of the rolling horizon: 2, 5, and 10. Next, we generate 10 instances of the average used-cash deposits and fit-cash demands throughout the T-period horizon. Thus, we have 10 vectors $(\bar{d}_t^u, \bar{d}_t^f), t = 1, 2, \ldots, T$. For each of these instances, we generate three instances of the demands/deposits intervals, corresponding to the widths of 5%, 10% and 15%. Thus, we have 30 vectors $(\kappa_t^u, \bar{d}_t^u, \nu_t^u, \kappa_t^f, \bar{d}_t^f, \nu_t^f), t = 1, 2, \ldots, T$. Finally, for each instance of the demands/deposits intervals, we generate 5 realized instances by randomly choosing the actual demands/deposits within the corresponding intervals, for a total of $30 \times 5 = 150$ instances of the T-period realized problem, P_a.

For each instance of P_a, let $\hat{X}^*$ denote its optimal solution and let $\hat{X}$ denote the solution obtained by Procedure ROLLING HORIZON described above. The percentage increase in cost of $\hat{X}$ over $\hat{X}^*$, computed as $\dfrac{(\text{Objective Value of } \hat{X} \text{ - Objective Value of } \hat{X}^*)}{\text{Objective Value of } \hat{X}^*} \times 100\%$, is an appropriate metric to capture the performance of Procedure ROLLING HORIZON. Each entry in Table 3.2 indicates the average of this gap over the corresponding 50 instances of BM corresponding to each value of w.

Table 3.2. Impact of the Length of the Rolling Horizon (τ) and Variation Width (w) on the Performance of Procedure ROLLING HORIZON.

$\tau \downarrow w \rightarrow$	5%	10%	15%
2	1.00%	2.00%	2.95%
5	0.76%	1.77%	2.75%
10	0.75%	1.77%	2.75%

For RM, we note that CPLEX (version 11.1.1) is not able to solve instances with $T = 100$ to optimality within a reasonable amount of time (2 hours of CPU time). Therefore, to compare the solutions obtained by our procedure, we used the best solution available from CPLEX. Considering the size of our test bed, we imposed a time limit of 20 minutes within CPLEX for each 100-period instance of RM. In most cases, the best solution available at the end of the imposed time limit is within 10% of the optimal. We then compare this solution with the one obtained by Procedure ROLLING HORIZON. Again, the average percentage gap between the two solutions is similar to that shown in Table 3.2.

Given that the optimal solutions to P_a are "ideal" – in the sense that they assume future data is known with certainty a priori – we observe that Procedure ROLLING HORIZON offers near-optimal, real-time solutions efficiently even in the presence of a considerable amount of variation (signified by a width of 15%). We conclude that practitioners can effectively use the two models (BM and RM), together with this procedure to plan the decisions of transportation, fit-sorting, and custodial inventory.

Since DIs need to manage cash inventory in the presence of the Fed's implemented policy, Problems BM and RM are indeed the real problems that DIs need to address and have, therefore, been the focus of our analysis. In Dawande et al. (2010), the authors argue that the Fed's policy may not induce DIs to behave in a socially-optimal manner. In the next section, we compare a DI's inventory decisions under the Fed's mechanism with those under a socially-optimal mechanism. From a DI's individual perspective, a socially-optimal mechanism may not be necessarily better than the Fed's mechanism. Thus, in addition to help us compare – from a DI's viewpoint – the relative benefits of the two mechanisms, such an analysis may also help the Fed fine-tune the parameters of its policy in the future.

3.6 The Fed's Policy and a Socially-Optimal Mechanism: A Comparative Analysis

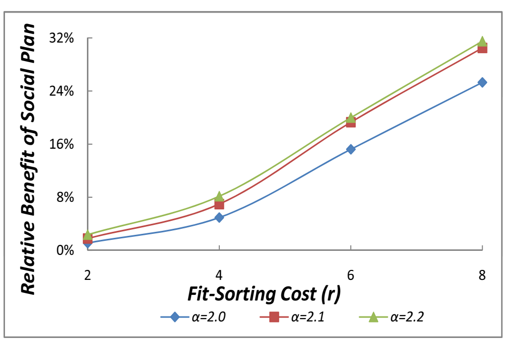
The Fed's mechanism focuses on cross-shipping with the intention of lessening its fit-sorting expense and, consequently, the societal cost of providing cash to the public. Dawande et al. (2010) argue that instead of this indirect manner of reducing social cost, the Fed can directly focus on reducing its fit-sorting volume, which, in turn, is defined by the non-fit-sorted used cash deposited to the Fed by the DIs. The authors propose a (hypothetical) socially-optimal mechanism that differs from the Fed's current policy in two respects: (1) the calculation of the volume of cross-shipping and (2) a reward for additional fit-sorting. The definition of cross-shipping is the same under both mechanisms. Under the Fed's policy, the volume of cross-shipping is calculated as the lesser of the fit cash in the used-cash deposit and the fit-cash withdrawal in the same week. In contrast, under the socially-optimal mechanism, the volume of cross-shipping is the amount of used-cash deposit, if there is also a fit-cash withdrawal in the same week. Also, under the socially-optimal mechanism, the DI receives a reward from the Fed if it deposits extra fit-sorted cash to the Fed after satisfying its own demand of fit cash.

We let RM^s denote the variant (of RM) that minimizes a DI's total cost incurred in managing the inventory of cash under the socially-optimal mechanism of Dawande et al (2010). The formulation of RM^s as an MIP is similar to that of RM in Section 3.3.3; we, therefore, avoid specifying the entire formulation and only summarize the main differences. In addition to the parameters of RM, the formulation of RM^s requires a new parameter w, defined as the ratio between the reward for depositing one bundle of fit-sorted cash and the fit-sorting cost (r/ρ) incurred for generating one bundle of fit-sorted cash. Furthermore, objective function of RM^s is to minimize the *net* cost for a DI, which is the difference between the total costs incurred (see Section 3.3.1) and the total reward received, over the planning horizon. Except for the constraints that define the volume of cross-shipping and the reward, all the other constraints in RM^s are the same as in RM. As in the case with RM, CPLEX was able to optimally solve the MIPs corresponding to the instances of RM^s in a reasonable time.

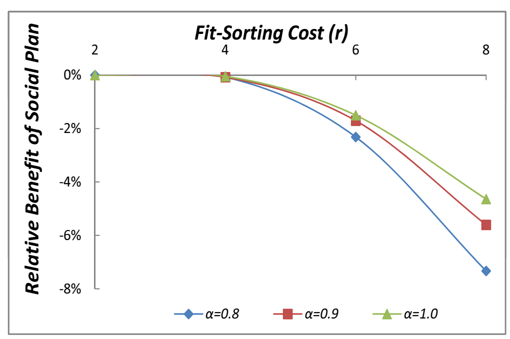
To illustrate the differences between the Fed's policy and the socially-optimal mechanism, we define the **Relative Benefit of Social Plan** (RBoSP) as the percentage cost saving offered by RM^s over RM. RBoSP captures the relative monetary impact on a DI from operating under the socially-optimal mechanism instead of the Fed's current policy. That is,

$$\text{RBoSP} = \frac{(\text{Objective Value of RM - Objective Value of RM}^s) \times 100\%}{\text{Objective Value of RM}}.$$

To understand the behavior of the RBoSP, it is instructive to examine the two mechanisms under two extreme scenarios: one in which both the ratio w and the imbalance ratio α are high and the other in which both these ratios are (relatively) low. To generate an appropriate test bed, we fix the length of the horizon (T) at 10. Also, the per bundle cross-shipping fee (e) and the per bundle per period holding cost (h) are fixed at \$5 and \$6, respectively. The per bundle fit-sorting cost (r) takes one of four values: 2, 4, 6, 8. Except for w and α, we keep the remaining parameters at their baseline values described in Section 3.4.1. Different combinations of the pair (w, α) are considered in the two scenarios. For each value of α, by using the approach described in Section 3.4.1, we generate 10 vectors of fit-cash demands and used-cash deposits over the horizon. Thus, for each value of α, we have 10 instances each of RM and RM^s. Figure 3.4(a) corresponds to the scenario when $w = 2$ and $\alpha \in \{2.0, 2.1, 2.2\}$ while Figure 3.4(b) represents the scenario when $w = 1$ and $\alpha \in \{0.8, 0.9, 1.0\}$. Each point in Figure 3.4 represents the average value of RBoSP over the 10 instances that correspond to a specific choice of α and r.



(a)



(b)

Figure 3.4. The Relative Benefit of Social Plan under Two Scenarios.

As seen in Figure 3.4(a), the RBoSP is non-negative and increases with the fit-sorting cost (r). Thus, in the first scenario, the DI incurs a lower cost under the socially-optimal policy as compared to that under the Fed's policy, and the gap between these two costs increases with the fit-sorting cost. This behavior can be explained using two properties: availability of used cash to fit-sort an additional amount and the reward for undertaking additional fit-sorting. Since the imbalance ratio $\alpha \geq 2$, the DI has the opportunity to fit-sort more, after satisfying its own demand of fit-cash. Also, since $w = 2$, fit-sorting extra used-cash is profitable for the DI. Consequently, relative to the Fed's mechanism, the DI fit-sorts more and cross-ships less under the socially-optimal mechanism, which, in turn, implies that it earns more reward and pays less penalty. Recall that w is the ratio of the reward for depositing one bundle of fit-sorted cash to the fit-sorting cost r/ρ incurred for generating one bundle of fit-sorted cash. Since w is fixed, the per-bundle reward increases with r. Thus, the relative benefit of operating under the socially-optimal mechanism progressively increases with r, reflecting in an increase in RBoSP.

In contrast, the second scenario results in Figure 3.4(b), where the RBoSP is non-positive and further decreases with r. Thus, the DI is worse off under the socially-optimal mechanism. We discuss the reward first. Since $\alpha \leq 1$, the DI cannot satisfy all of its fit-cash demand by fit-sorting its used-cash deposit. Also, $w = 1$ implies that the reward from depositing fit-sorted cash to the Fed is only enough to cover the fit-sorting expense incurred to generate it. Thus, the DI cannot derive significant benefit from the reward component of the socially-optimal mechanism. Next, we examine the difference in the cross-shipping penalty between the two mechanisms. When r is low, under both the mechanisms, the DI chooses to fit-sort most of the used-cash to avoid cross-shipping. In turn, the absence of any cross-shipping implies that there is no or little difference between the objective values of RM and RM^s.

When r increases, the DI prefers to fit-sort less in RM. Consequently, the DI has more used-cash inventory and needs to withdraw more fit-cash from the Fed. Thus, in general, the volume of cross-shipping increases. Note, however, that for the same solution, the DI is charged more cross-shipping penalty under the socially-optimal mechanism than under the Fed's policy. Thus, when r increases, the objective value of RM^s increases faster than that of RM; consequently, RBoSP decreases with r.

To summarize, *the DI can be either better or worse off under the socially-optimal mechanism than under the Fed's policy. There are two main determinants of the sign of RBoSP: the imbalance ratio* (α) *and the reward ratio* (w)*. In general, under the socially-optimal mechanism, when* α *and* w *are both large, the DI benefits from the reward component and is able to reduce its net cost to a value lower than that under the Fed's mechanism. When* α *and* w *are both small, the DI incurs a heavier penalty as compared to the Fed's mechanism and, thus, incurs a relatively higher net cost. In both cases, the percentage gap between the two mechanisms enlarges with an increase in* r.

3.7 Conclusions

The Federal Reserve's 2007 recirculation policy for physical cash is designed to incentivize DIs to locally reuse cash by fit-sorting their used-cash deposits. In this paper, we address the management of the inventories of different types of cash – used, fit, and unfit – over a finite planning horizon for a medium-size DI. The Fed's policy poses an important choice for a DI: whether or not to undertake the fit-sorting activity. Our first model (BM) captures the scenario in which a DI chooses not to fit-sort used cash. We identify two characteristics of an optimal solution regarding the withdrawals of fit cash and deposits of used cash. By exploiting these structural properties, we develop a polynomial-time dynamic programming algorithm. Our second model (RM) captures the situation when a DI undertakes fit-sorting (and is therefore also able to exploit custodial inventory, which is another component of the Fed's policy). We establish the hardness of this model and develop an integer programming formulation. Through a comprehensive numerical study, we conclude that the saving for a DI from implementing fit-sorting and custodial inventory as well as the extent of local recirculation are substantial. Thus, the Fed's eventual goal of reducing the social cost of providing currency to the public can be furthered. Moreover, the saving reaches its highest value when the imbalance between the used-cash deposit and fit-cash demand is just enough to effectively satisfy all demand via local reuse. To adapt our models for real-time decisions, we develop a rolling-horizon procedure which operates in an iterative decision-making environment. We show that our procedure provides near-optimal, real-time solutions to arbitrary realized instances when the widths of the interval forecasts over the planning horizon are modest. We also show that a DI can be either better or worse off under the Fed's implemented policy as compared to a (hypothetical) socially-optimal mechanism.

Limitations and Directions for Future Work

The two models – BM and RM – studied in this paper assume that data is known with certainty. While this simplifying assumption seems to be reasonable based on our interactions with practitioners, uncertainty may arise from three sources: demand of fit cash, deposit of used cash, and the proportion of fit cash in used-cash deposit. The rolling-horizon procedure in Section 3.5 performs well under the assumption that the widths of the interval forecasts over the planning horizon are modest. This can be seen as an effort to address – to a limited extent – the uncertainty from the first two sources. Nevertheless, the applicability of our models would benefit from a stochastic analysis in the presence of these uncertainties.

If cross-shipping cost is ignored, then the stochastic model corresponding to BM is equivalent to the problem of managing the inventories of two products experiencing stochastic demand and having a joint set-up cost. A limited set of results for this problem are available in the literature; see, e.g., Johnson (1967), Wheeler (1968), Kalin (1980), Liu and Esogbue (1999), Gallego and Sethi (2005). However, a characterization of the optimal policy under a general setting has, thus far, remained elusive. A promising but challenging future direction is to completely characterize the optimal policy for this problem. The main challenge here is that there is a fixed cost shared by two products. Thus, this direction is likely to warrant an advancement in the theory of multidimensional K-convexity. When cross-shipping cost is considered, the stochastic problems corresponding to both BM and RM involve complicated interactions between fit-cash withdrawals, used-cash deposits, and unfit-cash deposits. To our knowledge, such models have not been addressed in the literature. While it is unlikely that simple optimal policies exist for these problems, the task of obtaining near-optimal policies which can be efficiently implemented is a challenging and useful direction for future work.

It is instructive to establish a broader connection for the setting of our analysis in this paper. Conceptually, the need for local reuse in the cash supply chain is similar to that in the supply chain of false-failure returns (see, e.g., Ferguson et al. 2006). False-failure returns refer to those goods that are returned to the retailer (by customers) and, subsequently, to the manufacturer (by the retailer) but have no functional or cosmetic defects. Like used cash, a significant fraction of which is actually reusable, the manufacturer may find it economical to incentivize retailers to locally identify and recirculate faulty returns. In turn, the retailer may hire the services of a third-party provider to accomplish this task. Such a setting provides an opportunity to expand the analysis to a genuine multi-product domain.

CHAPTER 4

PROCESS INNOVATION VIA AN INDUSTRIAL SYMBIOTIC SYSTEM: THE IMPACT OF COMPETITION ON THE WILLINGNESS TO IMPLEMENT

4.1 Synopsis

The principle of industrial ecology (Allenby, 1992; Jellinski et al., 1992; Ehrenfeld, 1995) proposes to reorganize the industrial system so that it evolves towards a mode of operation that is compatible with the biosphere and is sustainable over the long term. Industrial ecology suggests the idea of an industrial food chain in which companies can be linked in some form of network, in order to exploit unutilized resources or by-products and thereby increase resource utilization. The two main elements under this concept are (1) the need to optimize the use of materials and energy, and (2) to close material loops while minimizing emissions.

Industrial Symbiosis, a subfield of industrial ecology, can be defined as engaging "traditionally separate industries in a collective approach to competitive advantage involving physical exchange of material, energy, water, and by-products" (Chertow, 2000). The term *By-Product Synergy* (BPS) has also been used synonymously in the literature with industrial symbiosis. BPS can offer true business opportunities beyond cost reduction, if wastes are viewed as raw materials for other industries. As BPS networks develop, industry goals may shift from reducing waste generation towards producing near-zero waste and finally to producing 100% product, while emissions are lowered and energy use is minimized (Mangan and

Olivetti 2010). The economic activity created in a BPS network creates new businesses and jobs, where the premise is that turning waste output from one organization into a product stream for another organization can generate revenue while reducing emissions and the need for virgin materials.

To focus on the operational issues involved in the implementation of an industrial symbiotic system, it is beneficial to first discuss a specific example.

4.1.1 A Real-World Implementation of a Symbiotic System

Seshasayee Paper and Boards (SPB; http://www.spbltd.com) was set up as a public enterprise in 1961 in the southern state of Tamil Nadu, India (see Private Communications listed at the end of the chapter). The company's main products include Bristol boards, Manila boards, colored bank paper, colored poster paper, and writing paper. The paper industry is capital intensive and highly dependent on easy access to high-quality raw materials. In particular, the paper industry in India has traditionally faced considerable difficulty with the availability of raw materials. The continued efforts by the Government of India to minimize de-forestation have progressively increased the scarcity of wood. Consequently, paper manufacturers have been consistently forced to look for alternate sources of raw materials. One such alternative for the paper production process is *bagasse*, a fibrous mass remaining after the extraction process of juice from sugarcane. The use of bagasse in paper production is energy efficient and also has a lesser impact on the environment, relative to wood. Traditionally, bagasse has been used as an important fuel input in the sugar industry and can meet the requirement of fuel for the industry. Since alternative fuels such as coal or furnace oil are relatively costlier, the sugar industry has been reluctant to sell the bagasse to the paper industry. Not surprisingly, following the steep increase in the prices of furnace oil and coal over the years, SPB was unable to make any arrangement to obtain a regular supply of bagasse from the sugar industry. The company also faces several environmental challenges. In addition to the emission of non-condensable gases, a paper mill also releases chlorinated compounds, dioxins, and furans. The waste water (effluent) carries high levels of Biochemical Oxygen Demand, Chemical Oxygen Demand, and other suspended solids. Furthermore, the problem of solid waste disposal is also a major concern in the face of local environmental pressures.

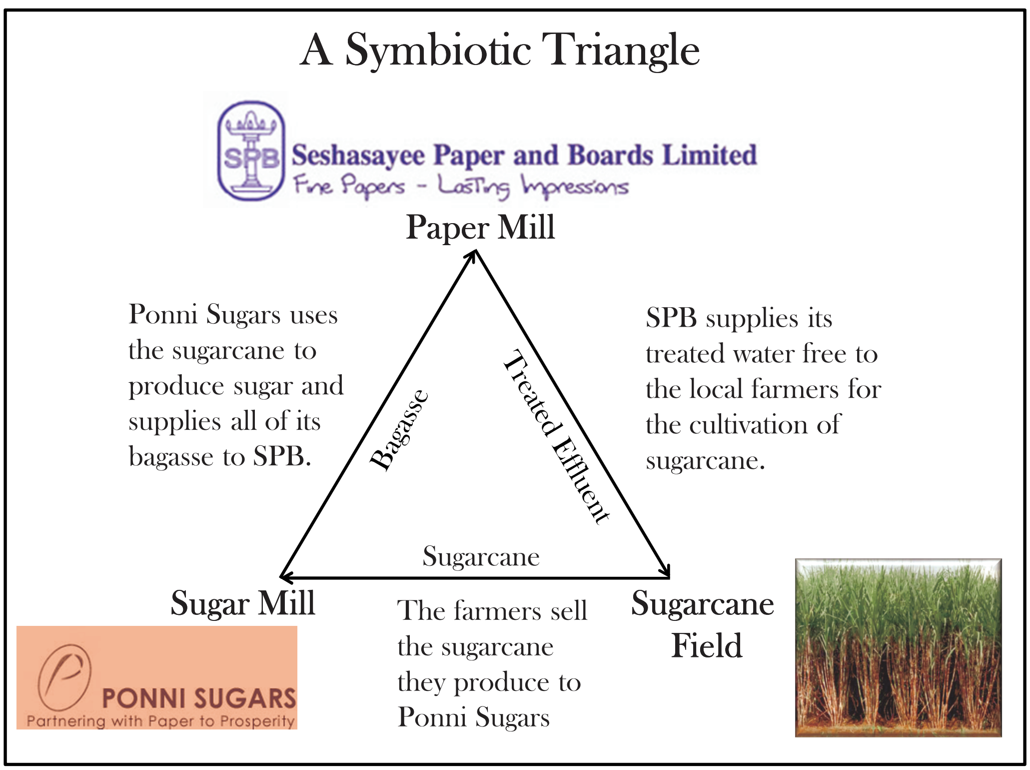


Figure 4.1. A Pictorial Illustration of the Paper-Sugar Symbiotic Industrial Complex.

SPB responded to these challenges by creating a revolutionary industrial complex. The company got involved in sugar production by locating a sugar mill (Ponni Sugars) close to its paper production facility, so that the entire output of bagasse from the production of sugar can be used as an input for the production of paper. This, however, led to another hurdle – availability of sugarcane that is a key ingredient for sugar production. This was a difficult problem as there was very little cultivation of sugarcane in the immediate neighborhood due to poor availability of water. The land around the nearby river Cauvery was elevated and dry, and there was a lack of facilities to pump water from the river (20 feet lower) to the surrounding areas. In 1991, the company entered into a tripartite agreement with the local farmers and its sugar mill, and decided to irrigate about 1500 acres of dry land with SPB's treated effluent for the cultivation of sugarcane. The sugar mill, in turn, purchases the sugarcane from these farmers and supplies its by-product bagasse to the paper mill of SPB. This "symbiotic triangle" is illustrated in Figure 4.1.

4.1.2 Differences from Traditional Process Innovation

There are several features of this implementation of the paper-sugar industrial complex that distinguish it from a traditional process innovation. We discuss a few below. Later, in Sections 4.2 and 4.3, our models exploit these properties.

1. *Simultaneous Change in the Production Processes of Multiple Products*: It is important to note the fundamental, and more environment-friendly, change in the use of bagasse in the symbiotic system. While bagasse has been used by sugar mills as a source of fuel, its use in paper production (in the symbiotic system) is as a core raw material. This

affects the production processes of both sugar and paper. On the one hand, the sugar mill needs to source alternate fuel since bagasse is no longer available. On the other hand, the production process of paper needs to be appropriately altered to use bagasse instead of wood. Also, in the larger symbiotic system, there are other implications: (1) sugar production benefits from the use of reliable and relatively cheap supply of sugarcane from the local farmers, (2) the effluent from paper production needs to be treated and delivered to the farmers, which is an additional expense for the paper mill, and (3) after implementation, the need to treat the waste from the productions of both paper and sugar reduces significantly relative to that before implementation. Clearly, these changes together imply advantages as well as disadvantages for both production processes.

2. *A Common Fixed Cost*: Since the process changes are symbiotic, the fixed investments required for the implementation naturally constitute one common cost for the entire implementation. This common fixed cost includes (i) acquiring the capability for the large-scale processing of bagasse in paper production, (ii) acquiring the capability to appropriately treat the effluent from the production processes, so that it can be used by sugarcane farmers, and (iii) building the transportation infrastructure for distributing the treated effluent to the sugarcane farmers.

3. *Linked Production Costs and Production Quantities*: Using the output (bagasse) of the sugar mill as raw material reduces the procurement cost of paper production. Therefore, the production quantities and marginal costs of the two products get interconnected. While the production quantities of paper and sugar do not necessarily

depend on each other (since, if required, the paper mill can obtain wood and the sugar mill can get additional sugarcane from other sources), their reciprocal influences on the cost economies in the symbiotic system are clear.

Finally, it should be emphasized that implementing the symbiotic system is a decision that implies accepting the (different) impacts on both the products together. In other words, it is not possible to change the production process of only one product while leaving the other unaffected. It is also clear from the discussion above that the cost-benefit tradeoff for the symbiotic system as a whole is not straightforward.

There are several other implementations of symbiotic systems that have been reported in the literature. We briefly mention three examples.

- Londonderry Eco-Industrial Park, New Hampshire (Block, 1998): The 100-acre park near the New Hampshire airport is one of the nation's prime examples of eco-industrial synergies. A plastics recycling company (a tenant at the park) purchases waste plastic from Stonyfield Farms Yogurt (a firm that is located close to the park). AES Corporation (Puerto Rico) commissioned and built a 720 MW combined natural gas power plant on site that uses treated waste water, which is pumped from the city of Manchester's sewage treatment facility, in its cooling towers.

- Industrial Symbiosis Park, Kalundborg, Denmark (Ehrenfeld and Gertler 1997): The park consists of a web of material and energy exchanges that occur amongst several diverse companies and the local community. There are five core partners – a power station, an oil refinery, a gypsum board facility, a pharmaceutical plant, and the city of Kalundborg – who have all developed a series of bilateral exchanges. The partners share ground water, surface water, waste water, steam, electricity, and also exchange a variety of residues that become feedstock for other processes.

- Monfort Boys Town Integrated BioSystem, Suva, Fiji (Klee and Williams, 1999; Bequette, 1997; Kane, 1997; Klee, 1999): This symbiotic system mixes agriculture and industry. Here, the waste from a brewery (spent grain) is used as a substrate to grow mushrooms, which are sold in the marketplace. The mushrooms break down the waste, making it a high-value, edible pig feed. The pigs are sold in the market, and the waste generated from the pigs is processed through an anaerobic digester. The treated waste is then piped to local fishponds, where the nutrient-rich water spawns food for several tropical layers of fish. The waste also creates fertile soil for growing vegetables.

4.1.3 Our Goals and Contributions

Our discussion thus far raises a variety of important research issues concerning symbiotic systems. This paper focuses on the following:

- **Characterization of the Impact of a Symbiotic System:** The differences with traditional process innovation (Section 4.1.2) imply non-trivial cost-side (operational) tradeoffs of implementing a symbiotic system. The implementation provides a firm an option of offering "green" (i.e., environment-friendly) variants of its products. Accordingly, the demand-side impact (to be soon discussed in Section 4.2.3) depends on the nature of the consumer population. Two other factors affecting the demand-side influence are the presence of a competing firm and the nature of the products offered by the competition. Motivated by the real-world system discussed in Section 4.1.1, we analyze a firm's production decisions for its two products – both in the presence and absence of a symbiotic system – under monopoly as well as under competition in Section 4.3.

- **Understanding the "Willingness" to Implement a Symbiotic System:** For a given setting (supply- and demand-side parameters; nature of competition, if any), the difference in a firm's total profits (from all the products connected by a symbiotic system) before and after the implementation of the system is a good measure to capture its willingness to implement the system. Section 4.4 exploits the corresponding optimal/equlibrium solutions to examine the behavior of this metric both under monopoly and competition, with respect to changes in several important parameters, including the variable production cost after implementation, consumers' appreciation of the green variant, and the proportion of green consumers.

- **Increased Willingness under Competition:** SPB's decision to implement the symbiotic system (Section 4.1.1) in 1991 came at an interesting time. Around the same time, the Government of India initiated the process of economic liberalization, which opened the country's markets to foreign firms in several sectors, including paper and pulp. This coincidence raises two interesting questions:

  1. Can the presence of competition increase the firm's willingness (relative to that under a monopoly) to implement the symbiotic system? We address this issue in Section 4.4.3 and identify some situations under which the answer is in the affirmative.

2. The firm can face competition either for its regular variant only or for both the regular and green variants. Can the willingness in the former situation be higher than that in the latter? Some conditions for an affirmative answer are identified in Section 4.4.3.

- **Benefiting Both the Firm and the Consumers:**

  While the benefits of a symbiotic system for the society at large are clear, a stronger motivation for the firm to implement the system results when both the firm and its consumers benefit. In Section 4.4.4, we identify some conditions under which such a simultaneous improvement occurs. We also illustrate a stronger case where a firm's willingness to implement is negative in a monopoly, but positive under competition and consumer welfare too increases after the implementation.

Technically, the challenges in our analysis are twofold: Under monopoly, we derive a firm's optimal prices of two variants (regular and green) in the presence of three types of consumers (regular, flexible green, and dedicated green) who each have different valuations for these two variants. It is important to emphasize that although the flexible green consumers have higher valuation for the green variant, they may choose to purchase the regular variant in case the price of the green variant is too high relative to that of the regular variant. Under Cournot competition, we derive the equilibrium production quantities – and the corresponding prices – of two firms for both the variants, under various settings of the nature of competition (competitors offering only the regular variants vs. both variants).

4.1.4 Literature Review

Three broad streams of literature are relevant to our paper – Closed-Loop Supply Chain Management, Industrial Ecology and Symbiotic Systems, and Process Innovation.

- **Closed-Loop Supply Chain Management:** The primary focus of the closed-loop supply chain literature has been on taking back products from consumers and recovering added value by using all or part of the products. Here, product recovery is addressed at the end of products' use or end of life with an emphasis on preventing them from entering the waste stream by using materials recovery systems and then adding value. Fleischmann et al. (1997), Guide et al. (2000), Guide and Jayaraman (2001) and Guide et al. (2003), all offer comprehensive reviews of reverse logistics and closed-loop supply chain research, including remanufacturing, recycling, reselling, and disposal. Most of these studies address a combination of cost-efficient recovery processes, while meeting prescribed environmental standards. Researchers including Toktay et al. (2000), Savaskan et al. (2004), Ferrer and Swaminathan (2006), and Geyer et al. (2007), study the implications of remanufacturing on supply chain logistics, operations, and product design, while Majumder and Groenevelt (2001) and Ferguson and Toktay (2006) study remanufacturing in a competitive setting.

  Recently, researchers have focused on the management of waste streams and the impact of regulation on both product and waste dispositions. For example, Subramanian et al. (2008) study the impact of extended producer responsibility for new product introductions in the electronics industry, while Plambeck and Wang (2009) investigate product design of durable goods in a sustainable supply chain environment. Additionally, Atasu et al. (2009), Esenduran and Kermahlioglu-Ziya (2009) and Esenduran et al. (2009) all study the operational impact of product take-back legislation. Guide and Van Wassenhove (2009) recount the evolution of research in the closed-loop supply chain area and discuss its development as a recognized subfield in supply chain operations.

The existing work on closed-loop supply chains primarily addresses post-consumer waste, where the source of the waste is the consumer. In contrast, our paper considers the simultaneous utilization of pre-consumer wastes from multiple products. We examine a setting where the waste from the manufacture of one product is used as a raw material for a second product, and vice-versa. In our setting, the firm proactively modifies its production processes to accomplish this goal.

- **Environmental Issues, Industrial Ecology, and Symbiotic Systems:** Environmental issues form an integral part of the broad framework of sustainability. As managers become increasingly aware of the long-term business implications of sustainability, organizations have moved beyond the consideration of whether or not it pays to be green (King and Lennox 2002; Walley and Whitehead 1994). The focus has now shifted on how to address environmental challenges, while maintaining competitiveness (Kleindorfer et al. 2005; Corbett and Klassen 2006). Studies that aim at integrating sustainability into environmental and economic systems include Allenby (1992), Jelinski et al. (1992), Allen and Behmanish (1994), Ehrenfeld and Gertler (1997), Korhonen (2004), and Wang et al. (2010). Researchers including Daly (1991) and Ehrenfeld

(1995) suggest simple design principles, including closing material loops, dematerialization, toxic elimination, and pollution prevention, that can serve as emerging models for operationalizing industrial ecology. Ehrenfeld and Gertler (1997) highlight industrial symbiosis via the Kalundborg case in Denmark, while Schwarz and Steininger (1997) and Posch et al. (1998) discuss the industrial recycling network at Styria. In both these cases, the key drivers are the potential cost reduction that is accomplished from waste avoidance and savings in virgin raw material. Lee (2011) studies a firm's operating strategies when it implements a by-product synergy on two products that are generated by a single production process.

In this paper, our attempt has been to characterize both the cost-side and demand-side implications of implementing an industrial symbiotic system. In particular, we show that these considerations generate rich tradeoffs and provide valuable insights on the decision to implement the environment-friendly initiative.

- **Process Innovation:** One consequence of a symbiotic system is the potential reduction in the production cost of one or more products that are manufactured in the system. In this sense, our work is related to the literature on "cost-reducing process innovations." Arrow (1962) considers the case of a firm undertaking a cost-reducing investment that cannot be imitated by competitors. He shows that the gain from a cost-reducing innovation is higher for a firm in a perfectly competitive industry than that for a monopolist. Bonanno and Haworth (1998) consider a vertically-differentiated industry to examine the question of whether cost-reducing innovations are more likely to be observed in regimes of more intense or less intense competition. Rosenkranz

(2003) studies the optimal division between product and process innovation under competition, when firms have an incentive to invest R&D activities in both. Martinez-Ros and Labeaga (2009) conclude that firms with a capability to engage in process innovation can be expected to generate higher profits, relative to those possible for non-innovators. Skea (1995) describes a range of process innovation strategies that organizations can adopt – from installing an end-of-pipeline technology (such as building a water treatment facility) to waste and emission-reduction strategies (such as process improvements through supply-chain redesign).

In our paper, a symbiotic system can be regarded as an unconventional approach for achieving by-product synergy, where multiple, simultaneous process innovations are implemented on the products linked by the system. Our analysis exploits this fundamental difference from traditional process innovation to offer insights on an organization's willingness to implement the system.

We now proceed to describe the setting of our analysis.

4.2 Model Setting

To address the research issues raised in Section 4.1.3, we develop several models to capture the operations of a domestic firm (Firm 1) that produces two products, P and S, under various settings. These models represent the scenarios both before and after the implementation of an industrial symbiotic system, and will be described in detail in Section 4.3. As justified later in Section 4.2.3, under an assumption that typically holds in practice, the firm's operational decisions for the two products can be made separately. Thus, for the purpose of analyzing the firm's optimal decisions (under monopoly) and equilibrium decisions (under competition), a single-product model suffices here. Subsequently, in Section 4.4, when evaluating the firm's strategic decision of whether or not to implement the symbiotic system, we consider the combined impact on the two products.

This section is organized as follows: in Section 4.2.1, we introduce the products and consumers studied in our models. Next, the structure of demand is described in Section 4.2.2. The cost-side and demand-side impacts of implementing the symbiotic system are formalized in Section 4.2.3.

4.2.1 Types of Product Variants and Consumers

There are two possible variants of the product: *regular* and *green*. If a firm does not implement the symbiotic system, its products can only be labeled as regular. Otherwise, if the system is implemented, then the firm has the option of labeling its products as green, which represents the use of an environment-friendly production process. Note that, in the latter case, the firm could still label its products as regular. We now introduce the primary notation used in our analysis. Additional notation is introduced later, as required.

Several surveys (e.g., Irland 1993, Forsyth et al. 1999) have indicated the existence of a significant proportion of environment-conscious consumers who are willing to pay a premium for variants produced by an ecologically friendly process. Wustenhagen et al. (2003) cite the example of Swiss utility companies that are able to offer "green" electricity by charging

Notation:

α	The upper bound of consumer's valuation of one unit of the regular variant.
a	The appreciation of the green variant, defined by the ratio of a green consumer's valuation for a unit of the green variant to that for a unit of the regular variant, $a \geq 1$.
p_r	The market price of the regular variant.
p_g	The market price of the green variant.
v_r	The consumer's valuation of the regular variant.
v_g	The consumer's valuation of the green variant.
Q_r	The total demand of the regular variant.
Q_g	The total demand of the green variant.

their customers premiums of about 400-700% over the regular product. Sedjo and Swallow (2002) model the premium for a green variant as a percentage over customers' valuation of the regular variant. We consider three types of consumers: *regular*, *flexible green* and *dedicated green*. On the one extreme, we have regular consumers who do not have any special appreciation for the "green" label. Accordingly, the valuation of these consumers for the green variant is the same as that for the regular variant. On the other extreme, dedicated green consumers are devoted to the green variant, if it is available. Thus, the valuation of dedicated green consumers for the regular variant becomes 0 if the green variant is available. In general, these consumers have a relatively higher valuation for the green variant. Flexible green consumers represent the intermediate kind: they simply have a higher valuation for the green variant, relative to the regular variant. These consumers buy the regular variant, if it results in a higher utility. The precise valuations of these three types of consumers are defined in Table 4.1.

We assume that an individual consumer's valuation for a unit of regular variant is uniformly distributed between 0 and an upper bound α, with density 1[1]. At any point v, $0 \leq v \leq$

[1]Note that there is no technical difficulty in assuming the density to be an arbitrary positive

Table 4.1. Valuations of the Three Types of Consumers for Regular and Green Variants, for an Arbitrary Value of v, $0 \leq v \leq \alpha$.

Consumer's Type ↓ Valuation →	Regular Variant, Green Variant (v_r, v_g)
Regular	(v, v)
Flexible Green	(v, av)
Dedicated Green	$(v, -)$, if the green variant is unavailable
	$(0, av)$, if the green variant is available

α, there stands one customer. With probability θ_1 (resp., θ_2 and θ_3), the consumer is a regular consumer (resp., a flexible green consumer and a dedicated green consumer). The distribution of consumers' valuation is shown in Figure 4.2.

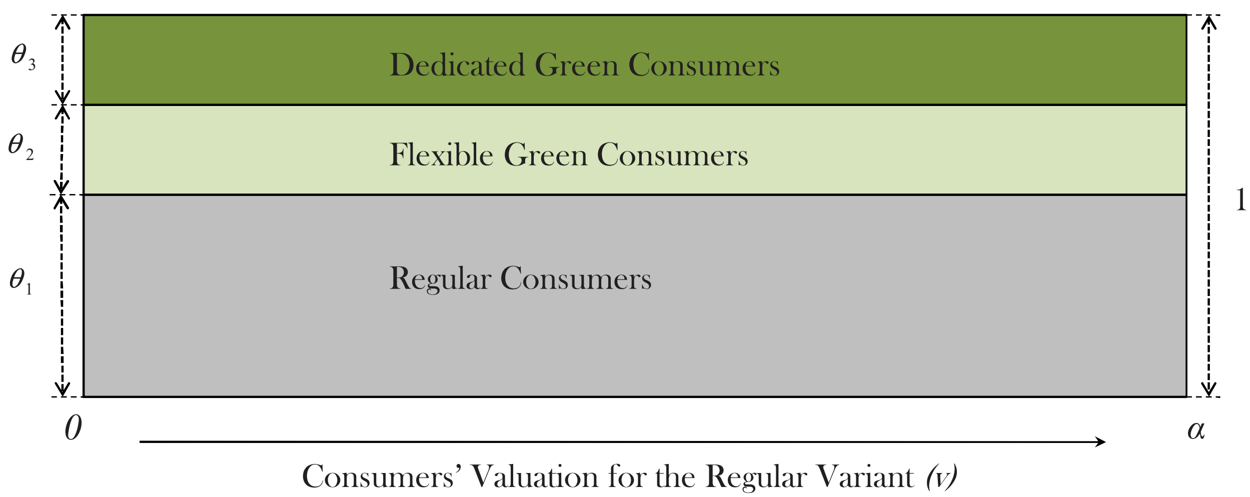


Figure 4.2. Distribution of Consumers' Valuation for the Regular Variant.

4.2.2 Structure of Demand

To obtain the aggregate demands of the regular variant and the green variant, we need to derive the demand of each type (regular/green) of variant from each type (regular/flexible green/dedicated green) of consumer. We now discuss these demands under three scenarios.

number, say n, other than 1. In this case, however, the value of n plays a role in the calculation of the firm's profit.

1. Only the regular variant is available.

   All the consumers who have a valuation higher than or equal to the market price will purchase the variant. Thus, we have

$$Q_r = \alpha - p_r.$$

2. Both regular and green variants are available.

   There are four possible price settings: $0 < p_g \leq p_r$, $p_r \leq p_g \leq ap_r$, $ap_r \leq p_g \leq (a-1)\alpha + p_r$, and $(a-1)\alpha + p_r \leq p_g \leq a\alpha$. The choices of each type of consumer and the total market demand for each variant under these four price settings are summarized in the following table. For brevity, we avoid providing the derivations of the expressions in the table.

Table 4.2. The Market Demand when Both Regular and Green Variants are Available.

	$\rightarrow$ *Price Setting* $\downarrow$ *Demand*	$0 < p_g \leq p_r$	$p_r \leq p_g \leq ap_r$	$ap_r \leq p_g \leq (a-1)\alpha + p_r$	$(a-1)\alpha + p_r \leq p_g \leq a\alpha$
Type(s) of Variants Chosen By Consumers	Regular Consumers	Green	Regular	Regular	Regular
	Flexible Green Consumers	Green	Green	Green and Regular	Regular
	Dedicated Green Consumers	Green	Green	Green	Green
Aggregate Demand	Total Demand of the Regular Variant (Q_r)	0	$(\alpha - p_r)\theta_1$	$(\alpha - p_r)\theta_1 + \frac{p_g - ap_r}{a-1}\theta_2$	$(\alpha - p_r)(\theta_1 + \theta_2)$
	Total Demand of the Green Variant (Q_g)	$(\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1-\theta_1)$	$(\alpha - \frac{p_g}{a})(1-\theta_1)$	$(\alpha - \frac{p_g}{a})\theta_3 + (\alpha - \frac{p_g - p_r}{a-1})\theta_2$	$(\alpha - \frac{p_g}{a})\theta_3$

3. Only the green variant is available.

   In this case, all the three types of consumers either buy the green variant or buy nothing. Thus,

$$Q_g = (\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

4.2.3 Cost-Side and Demand-Side Impacts of Implementing a Symbiotic System

Like traditional process innovation, the implementation of the symbiotic system affects a firm's production cost. Furthermore, it is clear from the discussion above that implementing the system also influences the market demand of a firm's variants. To describe these two impacts, we need some additional notation. For brevity, we only define the notation for Product P. The corresponding notation for Product S will then be clear.

Notation:

π_1	The profit of Firm 1 before the implementation of the symbiotic system.
$\pi_1^{'}$	The profit of Firm 1 after the implementation of the symbiotic system.
K	The fixed cost of implementing the symbiotic system.
p_p	The market price of Product P.
$q_{1,p}$	Firm 1's production quantity of Product P.
$b_1 q_{1,p}^2$	The waste treatment cost before the implementation of the symbiotic system.
$C_{1,p}(q_{1,p})$	The total production cost of Firm 1 to produce $q_{1,p}$ units of Product P prior to the implementation of the symbiotic system: $C_{1,p}(q_{1,p}) = b_1 q_{1,p}^2 + c_{1,p} q_{1,p}$, where b_1 and $c_{1,p}$ are positive constants, $b_1 \ll c_{1,p}$.
$b_1^{'} q_{1,p}$	The waste treatment cost after the implementation of the symbiotic system.
$C_{1,p}^{'}(q_{1,p})$	The total production cost of Firm 1 to produce $q_{1,p}$ units of Product P after the implementation of the symbiotic system: $C_{1,p}^{'}(q_{1,p}) = c_{1,p}^{'} q_{1,p}$, where $c_{1,p}^{'}$ is a positive constant whose value will be defined later.
β_p	The amount of waste generated from the production of one unit of Product P.
γ_p	The amount of waste of Product P needed to produce one unit of Product S.
s_p	The cost saving for using one unit of the waste of Product P to produce Product S.
$\bar{s}_p$	The revenue from selling one unit of the additional waste of Product P in the open market.

- **The Cost-Side Impact**

Before the implementation of a symbiotic system, Firm 1's total profit is:

$$\pi_1 = p_p q_{1,p} - b_1 q_{1,p}^2 - c_{1,p} q_{1,p} + p_s q_{1,s} - b_1 q_{1,s}^2 - c_{1,s} q_{1,s}.$$

The implementation of a symbiotic system affects the cost of production in three ways:

1. The additional fixed cost of implementing the symbiotic system

   The firm incurs an additional fixed cost K.

2. The savings in waste treatment cost

   We assume that the implementation simplifies the treatment process of the wastes of both the products, by reducing the need to extensively treat them. Thus, the waste treatment cost – which is represented by the quadratic term $(b_1 q_{1,p}^2)$ in the original production cost function – now has a linear form $(b_1^{'} q_{1,p})$.

3. The savings in procurement cost of raw material

   We assume that β_p units of waste are generated from the production of one unit of Product P. One unit of Product S can be produced by using γ_p units of this waste. The firm saves an amount of s_p for using every unit of the waste of Product P to produce Product S. Thus, the firm saves $(s_p \min\{\beta_p q_{1,p}, \gamma_p q_{1,s}\})$ from using the waste of Product P to produce Product S. If any waste of Product P remains after satisfying the required production of Product S, then we assume that the firm can sell it in the open market at a per-unit price $\bar{s}_p$; refer to Figure 4.3.

   Then, Firm 1's total profit after the implementation of the system is:

$$\begin{aligned}\pi_1^{'} &= p_p q_{1,p} - b_1^{'} q_{1,p} - c_{1,p} q_{1,p} + s_p \min\{\beta_p q_{1,p}, \gamma_p q_{1,s}\} + \bar{s}_p [\beta_p q_{1,p} - \gamma_p q_{1,s}]^+ \\ &\quad + p_s q_{1,s} - b_1^{'} q_{1,s} - c_{1,s} q_{1,s} + s_s \min\{\beta_s q_{1,s}, \gamma_s q_{1,p}\} + \bar{s}_s [\beta_s q_{1,s} - \gamma_s q_{1,p}]^+ - K.\end{aligned}$$

   Our analysis in this paper assumes that the difference between (i) the cost saving achieved by using one unit of the waste of Product P (resp., S) to produce Product S

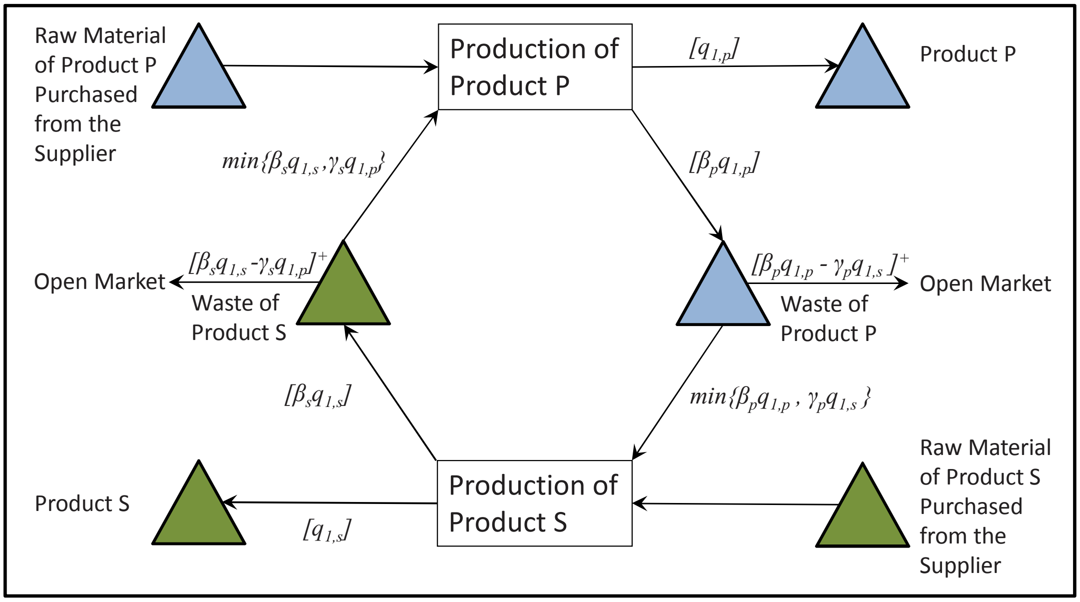


Figure 4.3. The Mutually Beneficial Use of Waste in a Symbiotic System. The Figure Shows the Production Cycle of Products P and S. The Notation is as Defined at the Start of Section 4.2.3.

(resp., P) and (ii) the revenue from selling one unit of the additional waste of Product P (resp., S), is negligible. We offer two arguments to justify this assumption in our context. First, in practical implementations, it is often the case that the waste generated from one product does not contribute a *dominant* supply of raw material required for the production of the other product. For instance, our extensive discussions with the senior administrators and operations managers at SPB (see Private Communications listed at the end of the paper) revealed that while a substantial amount of bagasse needed for paper comes from the sugar mill, it is far from sufficient to meet demand. Therefore, the company sources additional bagasse and wood from both domestic and international sources. Thus, all the waste of one product is used in the production of the other product, and vice-versa. Consequently, there is no additional waste of these products to be sold in the open market. This is a natural outcome in an "ideal" symbiotic system, since the useful utilization of all wastes is a cornerstone of the concept. So, in this case, we do not need the assumption. Second, even if additional waste remains, its quantity is typically small enough. Thus, there is sufficient demand in the open market for the firm to realize a per-unit revenue similar to the savings achieved if the waste were to be used for its own production. Under this assumption, Firm 1's total profit after implementation can be rewritten as follows:

$$\pi_1^{'} = p_p q_{1,p} - b_1^{'} q_{1,p} - c_{1,p} q_{1,p} + s_p \beta_p q_{1,p} + p_s q_{1,s} - b_1^{'} q_{1,s} - c_{1,s} q_{1,s} + s_s \beta_s q_{1,s} - K.$$

Denote $c_{1,p}^{'} = b_1^{'} + c_{1,p} - s_p \beta_p$, $c_{1,s}^{'} = b_1^{'} + c_{1,s} - s_s \beta_s$. Then, we have

$$\pi_1^{'} = p_p q_{1,p} - c_{1,p}^{'} q_{1,p} + p_s q_{1,s} - c_{1,s}^{'} q_{1,s} - K.$$

Note that depending on the value of $b_1^{'} - s_p \beta_p$, the coefficient $c_{1,p}^{'}$ of the production cost after the implementation could be either higher or lower than $c_{1,p}$.

- **The Demand-Side Impact**

Prior to the implementation of the symbiotic system, the firm can only label Products P and S as regular. Implementation of the system enables the firm to introduce the "green" (environment-friendly) variants of the two products, in an attempt to capture the consumers who have a relatively higher valuation for these variants and take advantage of their higher willingness to pay. We emphasize that, if the firm so chooses, it can also label a green variant as regular. One of the key characteristics of the symbiotic system is that both Products P and S can be labeled as green *simultaneously*. This symbolizes the "mutually dependant" relationship of the two products. The reactions of the three types of consumers (regular, flexible green, and dedicated green) to these two variants (see Section 4.2.2) then constitute the demand-side impact of the implementation.

In the next section, we analyze a firm's production decisions for the two products – both in the presence and absence of a symbiotic system – under monopoly as well as under competition.

4.3 Analyzing the Impact of the Symbiotic System under Monopoly and Competition

Section 4.3.1 considers the scenario in which the firm is the only supplier of both Products P and S in their respective markets. The justification of our chosen setting for analyzing competition and some related assumptions is provided in Section 4.3.2. Our discussion in Section 4.3.3 assumes that all green consumers are flexible. The case in which all green consumers are dedicated is discussed in Section 4.3.4. As argued in Section 4.2.3, the operational decisions about Product P and Product S can be made separately. Therefore, our discussion in this section is for a single product, say Product P. Accordingly, we simplify the notation by avoiding the subscript that represents the product (e.g., $c_{1,p}$ is simply denoted as c_1).

4.3.1 Monopoly Setting (Model M)

Consider the situation where Firm 1 is the only supplier of Product P in the market. We assume that the firm can decide the price of the product directly, and then produce to meet the generated demand. We also assume $c_1 < \alpha$ and $c_1^{'} < \alpha$. In Section 4.3.1, we discuss the scenario when the firm does not implement the symbiotic system and the scenario when the firm implements the symbiotic system.

No Symbiotic System For Firm 1

If Firm 1 does not implement the symbiotic system, then it can only produce the regular variant. Since the firm is the only supplier, only the regular variant is available in the market. The optimal price, obtained by solving the firm's profit maximization problem, is given in Theorem 4.3.1. Since this is a standard result under a monopoly setting, we avoid providing the proof.

Theorem 4.3.1 *In Model M, if Firm 1 does not implement the symbiotic system, then the optimal price of the regular variant is* $p_r^* = \frac{(1+2b_1)\alpha+c_1}{2(1+b_1)}$.

Symbiotic System For Firm 1

In this case, the firm can produce both the regular and green variants. The optimal prices for the variants are summarized in the following result.

Theorem 4.3.2 *In Model M, if Firm 1 implements the symbiotic system, the optimal price of regular variant is* $p_r^* = \frac{\alpha+c_1^{'}}{2}$ *and the optimal price of green variant is* $p_g^* = \frac{a\alpha+c_1^{'}}{2}$.

Proof: We consider four scenarios.

Scenario M1: $0 < p_g \leq p_r$

Since Firm 1 charges a lower price for the green variant, all consumers choose to buy either a green variant or nothing. The aggregate market demand is as follows:

$$Q_r = 0, \quad Q_g = (\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

Thus, Firm 1 faces the following profit-maximization problem:

$$\max_{p_r, p_g} \quad \pi_1 = (p_g - c_1^{'})Q_g = (p_g - c_1^{'})[(\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1 - \theta_1)].$$

The optimal price and optimal profit are as follows:

$$p_r^* \geq p_g^* = \frac{a\alpha + (a-1)\theta_1 c_1^{'} + c_1^{'}}{2[(a-1)\theta_1 + 1]}, \pi_1^{M1} = \frac{[a\alpha - (a-1)\theta_1 c_1^{'} - c_1^{'}]^2}{4a[(a-1)\theta_1 + 1]}.$$

Scenario M2: $p_r \leq p_g \leq ap_r$

Under this scenario, regular consumers purchase either the regular variant or nothing. Meanwhile, flexible and dedicated green consumers purchase either the green variant or nothing. The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)\theta_1, \quad Q_g = (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

Thus, Firm 1 faces the following profit-maximization problem.

$$\begin{aligned} \max_{p_r, p_g} \quad \pi_1 &= (p_r - c_1^{'})Q_r + (p_g - c_1^{'})Q_g \\ &= (p_r - c_1^{'})(\alpha - p_r)\theta_1 + (p_g - c_1^{'})(\alpha - \frac{p_g}{a})(1 - \theta_1). \end{aligned}$$

The optimal solution is in Table 4.3.

Since $p_r^* = \frac{\alpha + c_1^{'}}{2}$, $p_g^* = \frac{a\alpha + c_1^{'}}{2}$, $a \geq 1$, we have $p_r^* \leq p_g^* \leq ap_r^*$. Thus, Firm 1's maximum profit under Scenario M2 is $\pi_1^{M2} = \frac{\theta_1(\alpha - c_1^{'})^2}{4} + \frac{(1-\theta_1)(a\alpha - c_1^{'})^2}{4a}$.

Table 4.3. The Optimal Results Under Scenario M2.

	Regular Market	Green Market
Q^*	$\frac{\theta_1(\alpha-c_1^{'})}{2}$	$\frac{(1-\theta_1)(a\alpha-c_1^{'})}{2a}$
p^*	$\frac{\alpha+c_1^{'}}{2}$	$\frac{a\alpha+c_1^{'}}{2}$
π_1^*	$\frac{\theta_1(\alpha-c_1^{'})^2}{4}$	$\frac{(1-\theta_1)(a\alpha-c_1^{'})^2}{4a}$

We compare the profits under scenarios M1 and M2.

$$\begin{aligned} \pi_1^{M2} - \pi_1^{M1} &= \frac{\theta_1(\alpha - c_1^{'})^2}{4} + \frac{(1-\theta_1)(a\alpha - c_1^{'})^2}{4a} - \frac{[a\alpha - (a-1)\theta_1 c_1^{'} - c_1^{'}]^2}{4a[(a-1)\theta_1 + 1]} \\ &= \frac{\theta_1(1-\theta_1)(a-1)^2\alpha^2}{4[(a-1)\theta_1 + 1]} > 0. \end{aligned}$$

Scenario M3: $(a-1)\alpha + p_r \leq p_g \leq a\alpha$

Under this scenario, regular and flexible green consumers purchase either the regular variant or nothing. Dedicated green consumers purchase either the green variant or nothing. The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)(\theta_1 + \theta_2), \quad Q_g = (\alpha - \frac{p_g}{a})\theta_3.$$

Thus, Firm 1 faces the following profit-maximization problem:

$$\begin{aligned} \max_{p_r, p_g} \quad \pi_1 &= (p_r - c_1^{'})Q_r + (p_g - c_1^{'})Q_g \\ &= (p_r - c_1^{'})(\alpha - p_r)(\theta_1 + \theta_2) + (p_g - c_1^{'})(\alpha - \frac{p_g}{a})\theta_3 \\ s.t. \qquad & (a-1)\alpha + p_r \leq p_g \leq a\alpha. \end{aligned}$$

We first solve the unconstrained problem; the corresponding optimal solution is as follows:

Clearly, Firm 1's maximum profit is bounded from above by that of the unconstrained problem. Thus,

$$\pi_1^{M3} \leq \frac{(\theta_1 + \theta_2)(\alpha - c_1^{'})^2}{4} + \frac{\theta_3(a\alpha - c_1^{'})^2}{4a}.$$

Table 4.4. Optimal Results for Firm 1's Unconstrained Problem, under Scenario M3.

	Regular Market	Green Market
Q^*	$\frac{(\theta_1+\theta_2)(\alpha-c_1^{'})}{2}$	$\frac{\theta_3(a\alpha-c_1^{'})}{2a}$
p^*	$\frac{\alpha+c_1^{'}}{2}$	$\frac{a\alpha+c_1^{'}}{2}$
π_1^*	$\frac{(\theta_1+\theta_2)(\alpha-c_1^{'})^2}{4}$	$\frac{\theta_3(a\alpha-c_1^{'})^2}{4a}$

We can then compare the profits under Scenarios M2 and M3.

$$\begin{aligned}
\pi_1^{M2} - \pi_1^{M3} &\geq \frac{\theta_1(\alpha - c_1^{'})^2}{4} + \frac{(1-\theta_1)(a\alpha - c_1^{'})^2}{4a} - \frac{(\theta_1+\theta_2)(\alpha - c_1^{'})^2}{4} - \frac{\theta_3(a\alpha - c_1^{'})^2}{4a} \\
&= \theta_2[\frac{(a\alpha - c_1^{'})^2}{4a} - \frac{(\alpha - c_1^{'})^2}{4}] \\
&= \frac{\theta_2(a-1)[a\alpha^2 - (c_1^{'})^2]}{4a} > 0.
\end{aligned}$$

Scenario M4: $ap_r \leq p_g \leq (a-1)\alpha + p_r$

Under this scenario, regular consumers purchase the regular variant or nothing and dedicated green consumers purchase the green variant or nothing. Among flexible green consumers, some buy the regular variant, some others buy the green variant, and the remaining buy nothing. The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)\theta_1 + \frac{p_g - ap_r}{a-1}\theta_2, \quad Q_g = (\alpha - \frac{p_g}{a})\theta_3 + (\alpha - \frac{p_g - p_r}{a-1})\theta_2.$$

The firm's total profit can be written as follows:

$$\begin{aligned}
\max_{p_r, p_g} \quad \pi_1 &= Q_r p_r + Q_g p_g - c_1^{'}(Q_r + Q_g) \\
&= -(\theta_1 + \frac{a\theta_2}{a-1})p_r^2 + \frac{2\theta_2}{a-1}p_g p_r + [\alpha\theta_1 + c_1^{'}(\theta_1+\theta_2)]p_r \\
&\quad -(\frac{\theta_3}{a} + \frac{\theta_2}{a-1})p_g^2 + [\alpha(\theta_2+\theta_3) + c_1^{'}\frac{\theta_3}{a}]p_g - \alpha c_1^{'} \\
s.t. \quad & p_g - ap_r \geq 0, \\
& (a-1)\alpha + p_r - p_g \geq 0.
\end{aligned}$$

The Lagrangean and the Karush-Kuhn-Tucker optimality conditions are:

$$\begin{aligned}
L(p_r, p_g) &= \pi_1 + \lambda_1(p_g - ap_r) + \lambda_2[(a-1)\alpha + p_r - p_g] \\
\frac{\partial L}{\partial p_r} &= -2(\theta_1 + \frac{a\theta_2}{a-1})p_r + \frac{2\theta_2}{a-1}p_g + [\alpha\theta_1 + c_1^{'}(\theta_1+\theta_2)] \\
&\quad -a\lambda_1 + \lambda_2 = 0 \\
\frac{\partial L}{\partial p_g} &= -2(\frac{\theta_3}{a} + \frac{\theta_2}{a-1})p_g + \frac{2\theta_2}{a-1}p_r + [\alpha(\theta_2+\theta_3) + c_1^{'}\frac{\theta_3}{a}] \\
&\quad +\lambda_1 - \lambda_2 = 0 \\
\lambda_1(p_g - ap_r) &= 0, \\
\lambda_2[(a-1)\alpha + p_r - p_g] &= 0.
\end{aligned}$$

- **Scenario M4.1:** $\lambda_1 = 0$ and $\lambda_2 = 0$

  Solving the above system, we have

$$p_g^* = \frac{a\alpha + c_1^{'}}{2}, \qquad p_r^* = \frac{\alpha + c_1^{'}}{2}.$$

  However, since $p_g^* - ap_r^* = \frac{(1-a)c_1^{'}}{2} < 0$, the solution is invalid.

- **Scenario M4.2:** $p_g = ap_r$

  We can rewrite the demand functions as follows:

$$Q_r = (\alpha - p_r)\theta_1, \quad Q_g = (\alpha - \frac{p_g}{a})(1-\theta_1).$$

  Thus, the objective function has the same form as in Scenario M2. However, the optimization problem here has an additional constraint $p_g = ap_r$. Thus, the optimal value obtained in Scenario M2 is at least as good as that under Scenario M4.2.

- **Scenario M4.3:** $p_g = (a-1)\alpha + p_r$

  We can rewrite the demand functions as follows:

  $$Q_r = (\alpha - p_r)(\theta_1 + \theta_2), \quad Q_g = (\alpha - \frac{p_g}{a})\theta_3.$$

  Thus, the objective function has the same form as in Scenario M3. However, the optimization problem here has an additional constraint $p_g = (a-1)\alpha + p_r$. Thus, the optimal value obtained in Scenario M3 is at least as good as that under Scenario M4.3.

Combining the analysis above, we can conclude that the decisions obtained under Scenario M2 are optimal. This completes the proof. ■

The argument in the proof of Theorem 4.3.2 shows that the optimal setting of the prices corresponds to the scenario when regular consumers buy the regular variant, while all the flexible and dedicated green consumers buy the green variant. We note the following two implications.

Corollary 4.3.3 *Under a monopoly, if Firm 1 implements the symbiotic system, then it will avoid charging a price for the green variant that is high enough to drive (some) flexible green consumers to the regular variant. Under optimal prices for the regular and green variants, no flexible green consumer switches to the regular variant.*

Consequently, under the optimal prices, the behavior of flexible green consumers and dedicated green consumers is the same.

Corollary 4.3.4 *Under a monopoly, green consumers' type (either flexible or dedicated) does not affect Firm 1's optimal decisions.*

Next, we discuss the situation when Firm 1 faces a competitor. We first justify our choice of the setting to analyze competition and then specify two assumptions on the production amounts in equilibrium.

4.3.2 The Challenges in the Model Setting under Competition

The first challenge is to choose the type of competition. There are two possible settings: price and quantity. If we assume that the two firms compete on price, then under the assumption of a homogenous product, the firm with a lower variable production cost sets an equilibrium price just below the other firm's variable production cost. Thus, the sales for the high-cost firm become 0, while the low-cost firm captures the entire market. This is hardly the case in practice. Therefore, we choose to use Cournot competition and let the firms compete on quantity. If a firm is the only one to provide a given variant, we assume that it can decide the price of that variant directly.

The next challenge is to determine the market price of the regular variant. In our setting, we have three types of consumers and two variants of the product. The flexible green consumers may purchase the regular variant, when the price of the green variant is high enough. Thus, the information about the prices of the two variants is needed to determine the demand from the flexible green consumers for the regular variant. However, under Cournot competition, the price of the regular variant is determined by the total demand of that variant. To break this deadlock, we make the following assumption: the market price of the regular variant is determined by the Cournot inverse demand function of the regular consumers.

The third challenge is to assign the flexible green consumers' demand of the regular variant (if any) to the firms. There are two general approaches: (1) both firms share this demand (either equally or based on market share), and (2) one firm (e.g., the powerful one) exclusively supplies this demand. We choose the latter approach. Note that there is no technical difficulty to pursue the former approach. When Firm 1 can produce the green variant and Firm 2 cannot, each firm decides its production quantity of the regular variant. The market price of the regular variant is determined by the combined production quantity of both firms. Thus, each firm only partially influences the market price of the regular variant. Meanwhile, Firm 1 decides the price of the green variant directly. Thus, Firm 1 is relatively more influential in deciding the market prices of the two variants. We emphasize this advantage by allocating the demand of the regular variant from the flexible green consumers to Firm 1.

To focus our analysis on realistic situations, we make the following two assumptions:

1. If a firm has the option to label its product as the green variant, then it will produce a positive amount of the green variant. Thus, the firm has two choices: (i) providing only the green variant, or (ii) providing both regular and green variants.

   The theoretically possible scenario, in which a firm provides only the regular variant even if it can label its product as green, is rarely seen in practice. A firm will naturally be reluctant to forgo the opportunity to exploit the more-profitable green market. Under Cournot competition, since the price of a variant is determined by the total production quantities of two firms, neither firm can directly decide the prices of both variants. Thus, a firm's action of providing both regular and green variants is legal from the viewpoint of anti-price discrimination regulations (e.g., the Robinson-Patman Act).

2. The total amount of the product (i.e., the sum of the regular and green variants of the product, as applicable) produced by each firm is positive.

   As mentioned earlier, we avoid considering the situation where a firm drives its competitor out of the market.

The Chosen Setting under Competition

We assume two firms, Firm 1 and Firm 2, in the market. Competition is realized via the following 2-stage game.

- Stage 1. Each firm indicates the types of variants it will produce. If a firm has the option of labeling its product as the green variant, then it has two choices: (1) providing only the green variant, or (2) providing both the regular and green variants. Each firm decides its choice and makes this decision public.

- Stage 2. If there are two firms who produce a given variant, then the firms decide their individual production quantities simultaneously. The market price of this variant is then determined by the corresponding Cournot demand function. If a firm is the only one to provide a given variant, then it decides the price of that variant directly.

For consistency, we continue to use the notation defined in Section 4.2. We need the following additional notation.

Notation:

q_2	The production quantity of Firm 2.
$C_2(q_2)$	The production cost of Firm 2 to produce q_2 units of product; $C_2(q_2) = c_2q_2$.
π_2	The profit of Firm 2.

Although equilibrium results can be obtained in the presence of both dedicated and flexible green consumers, (i.e., $\theta_2 > 0$, $\theta_3 > 0$), for the purpose of deriving managerial insights, we restrict our analysis to two special cases: (i) $\theta_2 = 0$ (all green consumers are dedicated), (ii) $\theta_3 = 0$ (all green consumers are flexible). The case when all green consumers are flexible is considered in Section 4.3.3. The case in which all green consumers are dedicated is analyzed in Section 4.3.4.

4.3.3 Presence of a Competitor When All Green Consumers Are Flexible

In this case, depending on the prices of the green and regular variants, the green consumers may choose to buy either one. As described in Section 4.2.2, consumers' demand for the regular and green variants has multiple possible forms. We focus our attention here on the more advantageous special case (for Firm 1) in which the competitor only produces the regular variant and Firm 1 decides to implement the symbiotic system.

Symbiotic System For Firm 1

Recall our assumption that both firms produce positive amounts of the product. Since Firm 2 can only produce the regular variant, a necessary condition for this to occur is that the price of the regular variant is lower than that of the green variant but higher than Firm 2's variable production cost ($c_2 < p_r^* < p_g^*$). If Firm 1's variable production cost after implementation is less than Firm 2's variable production cost ($c_1^{'} < c_2$), then the former is lower than the price of the regular variant as well ($c_1^{'} < p_r^*$). Thus, Firm 1 only producing the green variant is not an equilibrium, since Firm 1 can always improve its profit by providing a nonzero amount of the regular variant, and gain additional profit. Thus, in equilibrium, Firm 1 provides both regular and green variants, while Firm 2 only provides the regular variant. The following result summarizes the equilibrium prices under three possible parameter settings.

Theorem 4.3.5 *When $a > 1$ and $c_1^{'} < c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$, if Firm 1 implements the symbiotic system, then there are three possible equilibria (depending on parametric relationships). These are described below and pictorially illustrated in Figure 4.4.*

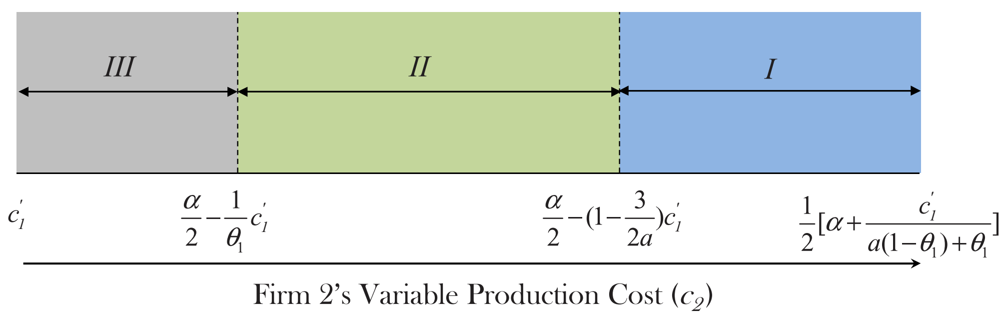


Figure 4.4. Categorization of Equilibria Based on the Competitor's Variable Production Cost (c_2), when $a > 1$, $c_1^{'} < \frac{\theta_1\alpha}{2(1+\theta_1)}$, and $c_1^{'} < c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$.

1. *If $c_1^{'} < \frac{\theta_1\alpha}{2(1+\theta_1)}$, then*

   - *Type I: If $[\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] < c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$, then we have*

   $$p_g^I = \frac{a\alpha + c_1^{'}}{2}, \quad q_{1,r}^I = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)}{3}, \quad q_{2,r}^I = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{3}.$$

   - *Type II: If $(\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) \leq c_2 \leq [\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}]$, then we have*

   $$p_g^{II} = \frac{a(a + \theta_1 - a\theta_1)\alpha + ac_1^{'} + a\theta_1 c_2}{2a(1-\theta_1) + 3\theta_1}, q_{1,r}^{II} = \frac{\theta_1[\theta_1\alpha - 2c_1^{'} + (2a - 2a\theta_1 + \theta_1)c_2]}{2a(1-\theta_1) + 3\theta_1},$$

   $$q_{2,r}^{II} = \frac{\theta_1[(a - a\theta_1 + \theta_1)\alpha + c_1^{'} - 2(a - a\theta_1 + \theta_1)c_2]}{2a(1-\theta_1) + 3\theta_1}.$$

   - *Type III: If $c_1^{'} < c_2 < (\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'})$, then we have*

   $$p_g^{III} = \frac{(2a + a\theta_1 - \theta_1)\alpha + 2c_1^{'} + 2\theta_1 c_2}{2(2 + \theta_1)},$$

$$q_{1,r}^{III} = \frac{\theta_1[\theta_1\alpha - 2c_1^{'} + (2-\theta_1)c_2]}{2+\theta_1}, \quad q_{2,r}^{III} = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{2+\theta_1}.$$

2. *If* $\frac{\theta_1\alpha}{2(1+\theta_1)} \leq c_1^{'} < \frac{a\alpha}{4a-3}$*, then the region representing Type III vanishes.*

3. *If* $\frac{a\alpha}{4a-3} \leq c_1^{'} < \frac{\alpha}{2-\frac{1}{a(1-\theta_1)+\theta_1}}$*, then the regions representing Type II and III vanish.*

Proof: In equilibrium, Firm 1 provides both regular and green variants. Firm 2 only provides the regular variant. Recall that among the multiple price settings described in Section 4.2.2, only $p_r < p_g \leq ap_r$ and $ap_r \leq p_g < (a-1)\alpha + p_r$ guarantee the existence of both regular and green variants in equilibrium. Thus, following our assumptions in Section 4.3.2, we consider only these two price settings in our analysis.

1. If $c_1^{'} < \frac{\theta_1\alpha}{2(1+\theta_1)}$

   We first consider the values of the lower and upper bounds of c_2. Since $a > 1$, $a(1-\theta_1)+\theta_1$ decreases with an increase in θ_1. Thus, for $0 \leq \theta_1 \leq 1$, we have $1 \leq a(1-\theta_1)+\theta_1 \leq a$. Therefore, we have $\frac{\alpha}{2} + \frac{c_1^{'}}{2a} \leq \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}] \leq \frac{\alpha}{2} + \frac{c_1^{'}}{2}$. Thus, $c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}] \leq (\frac{\alpha}{2} + \frac{c_1^{'}}{2})$.

   Since $a > 1$, $0 \leq \theta_1 \leq 1$, and $c_1^{'} < \frac{\theta_1\alpha}{2(1+\theta_1)}$, we have

$$(\frac{\alpha}{2} + \frac{c_1^{'}}{2a}) - [\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] = (1 - \frac{1}{a})c_1^{'} > 0,$$
$$[\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] - (\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) = (\frac{3}{2a} + \frac{1-\theta_1}{\theta_1})c_1^{'} > 0,$$
$$(\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) - c_1^{'} > \frac{\alpha}{2} - (1 + \frac{1}{\theta_1})\frac{\theta_1\alpha}{2(1+\theta_1)} = 0.$$

   Thus, $c_1^{'} < (\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) < [\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] < (\frac{\alpha}{2} + \frac{c_1^{'}}{2a}) \leq \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$. Therefore, all the three regions, which represent the corresponding three types of equilibria, have positive lengths.

We now consider three scenarios:

Scenario D1: $p_r < p_g < ap_r$

Under this scenario, regular consumers only purchase regular products and flexible green consumers only purchase green products. The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)\theta_1, \quad Q_g = (\alpha - \frac{p_g}{a})(1-\theta_1).$$

Thus, we have $p_r = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1}$. Firm 1's total profit can be written as follows:

$$\begin{aligned} \max_{q_{1r},p_g} \quad \pi_1 &= q_{1r}(p_r - c_1^{'}) + Q_g(p_g - c_1^{'}) \\ &= q_{1r}(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'}) + (\alpha - \frac{p_g}{a})(1-\theta_1)(p_g - c_1^{'}). \end{aligned}$$

The first-order conditions are

$$\frac{\partial \pi_1}{\partial q_{1r}} = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'} - \frac{q_{1r}}{\theta_1} = 0, \tag{4.1}$$

$$\frac{\partial \pi_1}{\partial p_g} = (\alpha - \frac{p_g}{a})(1-\theta_1) - \frac{(1-\theta_1)(p_g - c_1^{'})}{a} = 0. \tag{4.2}$$

Next we find the Hessian for $\pi_1(q_{1r}, p_g)$

$$H(q_{1r}, p_g) = \begin{bmatrix} -\frac{2}{\theta_1} & 0 \\ 0 & -\frac{2(1-\theta_1)}{a} \end{bmatrix}$$

Since $H_1(q_{1r}, p_g) = -\frac{2}{\theta_1} < 0$, $H_2(q_{1r}, p_g) = (-\frac{2}{\theta_1})(-\frac{2(1-\theta_1)}{a}) > 0$, the first-order conditions are both necessary and sufficient.

Firm 2's profit is $\max_{q_{2r}} \pi_2 = (p_r - c_2)q_{2r} = (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2)q_{2r}$. The first-order condition is

$$\frac{\partial \pi_2}{\partial q_{2r}} = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2 - \frac{q_{2r}}{\theta_1} = 0 \tag{4.3}$$

Solving (4.1), (4.2), and (4.3), we obtain the following solution:

	Regular Market	Green Market
q_1^*	$\frac{\theta_1(\alpha-2c_1^{'}+c_2)}{3}$	$\frac{(1-\theta_1)(a\alpha-c_1^{'})}{2a}$
q_2^*	$\frac{\theta_1(\alpha+c_1^{'}-2c_2)}{3}$	0
p^*	$\frac{\alpha+c_1^{'}+c_2}{3}$	$\frac{a\alpha+c_1^{'}}{2}$

Verifying Validity: We now verify that the quantities derived above are all positive. Since $c_1^{'} < c_2 < \frac{\alpha+c_1^{'}}{2}$, we have $\alpha - 2c_1^{'} + c_2 = (\alpha - c_1^{'}) + (c_2 - c_1^{'}) > 0$, and $\alpha + c_1^{'} - 2c_2 > 0$. Thus, $q_{1,r}^* > 0$ and $q_{2,r}^* > 0$. Also, since $c_1^{'} < \frac{\alpha+c_1^{'}}{2}$, we have $c_1^{'} < \alpha < a\alpha$. Thus, $q_{1,g}^* > 0$. We also need to verify that the optimal prices satisfy the constraint $p_r < p_g < ap_r$. We have $p_g - p_r = \frac{(3a-2)\alpha+c_1^{'}-2c_2}{6} \geq \frac{\alpha+c_1^{'}-2c_2}{6} > 0$. We also have $ap_r - p_g = \frac{-a\alpha+(2a-3)c_1^{'}+2ac_2}{6}$. Therefore, if $c_2 > \frac{\alpha}{2} - (1-\frac{3}{2a})c_1^{'}$, then we have $ap_r^* > p_g^*$. Thus, the above solution is valid.

Scenario D2: $p_g = ap_r$

The aggregate market demand is the same as that in scenario D1. We have $p_r = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1}$ and $p_g = ap_r$. Firm 1's total profit can be written as follows:

$$\begin{aligned}
\max_{q_{1r}} \quad \pi_1 &= q_{1r}(p_r - c_1^{'}) + Q_g(ap_g - c_1^{'}) \\
&= q_{1r}(p_r - c_1^{'}) + (\alpha - p_r)(1-\theta_1)(ap_r - c_1^{'}) \\
&= q_{1r}(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'}) + \frac{(q_{1r}+q_{2r})}{\theta_1}(1-\theta_1)[a(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1}) - c_1^{'}].
\end{aligned}$$

The first-order condition is

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_{1r}} &= \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'} - \frac{q_{1r}}{\theta_1} + \frac{(1-\theta_1)}{\theta_1}[a(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1}) - c_1^{'}] \\
&\quad - \frac{a(q_{1r}+q_{2r})}{\theta_1^2}(1-\theta_1) = 0. \qquad (4.4)
\end{aligned}$$

Firm 2's profit is $\max_{q_{2r}} \pi_2 = (p_r - c_2)q_{2r} = (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2)q_{2r}$. The first-order condition is

$$\frac{\partial \pi_2}{\partial q_{2r}} = \alpha - \frac{q_{1r} + q_{2r}}{\theta_1} - c_2 - \frac{q_{2r}}{\theta_1} = 0 \tag{4.5}$$

Solving (4.4) and (4.5), we obtain the following solution:

	Regular Market	Green Market
q_1^*	$\frac{\theta_1[\alpha\theta_1 - 2c_1^{'} + c_2(2a - 2a\theta_1 + \theta_1)]}{2a(1-\theta_1)+3\theta_1}$	$\frac{(1-\theta_1)[\alpha(a - a\theta_1 + 2\theta_1) - c_1^{'} - c_2\theta_1]}{2a(1-\theta_1)+3\theta_1}$
q_2^*	$\frac{\theta_1[\alpha(a - a\theta_1 + \theta_1) + c_1^{'} - 2c_2(a - a\theta_1 + \theta_1)]}{2a(1-\theta_1)+3\theta_1}$	0
p^*	$\frac{\alpha(a+\theta_1 - a\theta_1) + c_1^{'} + c_2\theta_1}{2a(1-\theta_1)+3\theta_1}$	$\frac{a[\alpha(a+\theta_1 - a\theta_1) + c_1^{'} + c_2\theta_1]}{2a(1-\theta_1)+3\theta_1}$

Verifying Validity: We now verify that the quantities derived above are all positive. Since $c_1^{'} < c_2$, we have $\alpha\theta_1 - 2c_1^{'} + c_2(2a - 2a\theta_1 + \theta_1) = \theta_1(\alpha - c_1^{'}) + (2-\theta_1)(c_2 - c_1^{'}) + 2(a-1)(1-\theta_1)c_2 > 0$. Thus, $q_{1,r}^* > 0$. Since $c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$, we have $\alpha(a - a\theta_1 + \theta_1) + c_1^{'} - 2c_2(a - a\theta_1 + \theta_1) > 0$. Thus, $q_{2,r}^* > 0$. Also, since $c_1^{'} < \alpha$ and $c_2 < \alpha$, we have $\alpha(a - a\theta_1 + 2\theta_1) - c_1^{'} - c_2\theta_1 > \alpha(a - a\theta_1 + 2\theta_1) - \alpha - \theta_1\alpha = \alpha(a-1)(1-\theta_1) \geq 0$. Thus, $q_{1,g}^* > 0$.

Scenario D3: $ap_r < p_g < (a-1)\alpha + p_r$

Under this scenario, regular consumers purchase the regular variant or nothing. Among flexible green consumers, some buy the regular variant, some others buy the green variant, and the remaining buy nothing. The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)\theta_1 + \frac{p_g - ap_r}{a-1}(1-\theta_1), \quad Q_g = (\alpha - \frac{p_g - p_r}{a-1})(1-\theta_1).$$

Recall our assumption (from Section 4.3.2) that the market price of the regular variant is determined by the Cournot inverse demand function of the regular consumers. Thus, we have $p_r = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1}$. Firm 1's total profit can be written as follows:

$$\begin{aligned}\max_{q_{1r},p_g} \quad \pi_1 &= [q_{1r} + \frac{p_g - ap_r}{a-1}(1-\theta_1)](p_r - c_1^{'}) + (\alpha - \frac{p_g - p_r}{a-1})(1-\theta_1)(p_g - c_1^{'}) \\ &= [q_{1r} + \frac{p_g - a(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1})}{a-1}(1-\theta_1)](\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'}) \\ &\quad +(\alpha - \frac{p_g - (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1})}{a-1})(1-\theta_1)(p_g - c_1^{'}).\end{aligned}$$

The firs-order conditions are

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_{1r}} &= [1 + \frac{a(1-\theta_1)}{(a-1)\theta_1}](\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'}) - \frac{1}{\theta_1}[q_{1r} + \\ &\quad \frac{p_g - a(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1})}{a-1}(1-\theta_1)] - \frac{(1-\theta_1)(p_g - c_1^{'})}{\theta_1(a-1)} = 0, \qquad (4.6)\\ \frac{\partial \pi_1}{\partial p_g} &= \frac{1-\theta_1}{a-1}(\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'}) + (\alpha - \frac{p_g - (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1})}{a-1})(1-\theta_1) \\ &\quad -\frac{1-\theta_1}{a-1}(p_g - c_1^{'}) = 0. \qquad (4.7)\end{aligned}$$

Next, we consider the Hessian for $\pi_1(q_{1r}, p_g)$

$$H(q_{1r}, p_g) = \begin{bmatrix} -\frac{2(a-\theta_1)}{\theta_1^2(a-1)} & -\frac{2(1-\theta_1)}{\theta_1(a-1)} \\ -\frac{2(1-\theta_1)}{\theta_1(a-1)} & -\frac{2(1-\theta_1)}{a-1} \end{bmatrix}$$

Since $H_1(q_{1r}, p_g) = -\frac{2(a-\theta_1)}{\theta_1^2(a-1)} < 0$, $H_2(q_{1r}, p_g) = \frac{4(1-\theta_1)}{\theta_1^2(a-1)} > 0$, the first-order conditions are both necessary and sufficient.

Firm 2's profit is

$$\begin{aligned}\max_{q_{2r}} \quad \pi_2 &= (p_r - c_2)q_{2r} \\ &= (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2)q_{2r}\end{aligned}$$

The first-order condition is

$$\frac{\partial \pi_2}{\partial q_{2r}} = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2 - \frac{q_{2r}}{\theta_1} = 0 \qquad (4.8)$$

Solving (4.6), (4.7), (4.8), we obtain the following solution:

	Regular Market	Green Market
q_1^*	$\frac{\theta_1[\alpha\theta_1-2c_1^{'}+c_2(2-\theta_1)]}{2+\theta_1}$	$\frac{\alpha(1-\theta_1)}{2}$
q_2^*	$\frac{\theta_1(\alpha+c_1^{'}-2c_2)}{2+\theta_1}$	0
p^*	$\frac{\alpha+c_1^{'}+c_2\theta_1}{2+\theta_1}$	$\frac{\alpha(2a+a\theta_1-\theta_1)+2c_1^{'}+2c_2\theta_1}{2(2+\theta_1)}$

Verifying Validity: Since $c_1^{'} < c_2$ and $c_1^{'} < \alpha$, we have $\alpha\theta_1 - 2c_1^{'} + c_2(2-\theta_1) = \theta_1(\alpha - c_1^{'}) + (2-\theta_1)(c_2 - c_1^{'}) > 0$. Thus, $q_{1,r}^* > 0$. Since $c_2 < (\frac{\alpha}{2} + \frac{c_1^{'}}{2})$, we have $q_{2,r}^* > 0$. Also, $q_{1,g}^* > 0$.

We also need to verify that the optimal prices satisfy the constraint $p_g \geq ap_r$. We have $p_g - ap_r = \frac{(a-1)(\theta_1\alpha-2c_1^{'}-2\theta_1c_2)}{2(2+\theta_1)}$. Thus, if $c_2 < (\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'})$, then we have $p_g - ap_r > 0$.

We consider the two interior solutions in scenario D1 and scenario D3 (given that each satisfies the corresponding constraint) and one boundary solution in scenario D2. The equilibrium prices in these three scenarios are categorized and validated as follows:

- Type I: If $p_r < p_g < ap_r$, we have

$$p_r^* = \frac{\alpha + c_1^{'} + c_2}{3}, p_g^* = \frac{a\alpha + c_1^{'}}{2}$$

Under the imposed condition, $[\frac{\alpha}{2} - (1 - \frac{3}{2a}c_1^{'})] < c_2 < \frac{1}{2}[\alpha + \frac{c_1^{'}}{a(1-\theta_1)+\theta_1}]$, we indeed have have $p_r^* < p_g^* < ap_r^*$.

- Type II: If $ap_r = p_g$, we have

$$p_r^* = \frac{(a+\theta_1-a\theta_1)\alpha + c_1^{'} + \theta_1 c_2}{2a(1-\theta_1)+3\theta_1}, p_g^* = \frac{a(a+\theta_1-a\theta_1)\alpha + ac_1^{'} + a\theta_1 c_2}{2a(1-\theta_1)+3\theta_1}$$

- Type III: If $ap_r < p_g < (a-1)\alpha + p_r$, we have

$$p_r^* = \frac{\alpha + c_1^{'} + \theta_1 c_2}{2+\theta_1}, p_g^* = \frac{(2a + a\theta_1 - \theta_1)\alpha + 2c_1^{'} + 2\theta_1 c_2}{2(2+\theta_1)}$$

  Under the imposed condition, $c_1^{'} < c_2 < (\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'})$, we have $ap_r^* < p_g^* < (a-1)\alpha + p_r^*$.

Since $[\frac{\alpha}{2} - (1 - \frac{3}{2a}c_1^{'})] > [\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}]$, Type I and Type III solutions cannot both be valid. When either of these two solutions is valid, it is straightforward to show that the valid solution is also better than the Type II solution. When neither is valid, the Type II solution is optimal.

2. If $\frac{\theta_1\alpha}{2(1+\theta_1)} \le c_1^{'} < \frac{a\alpha}{4a-3}$

   We have

$$\begin{aligned}
&(\frac{\alpha}{2} + \frac{c_1^{'}}{2a}) - [\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] = (1 - \frac{1}{a})c_1^{'} > 0, \\
&[\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] - c_1^{'} > \frac{\alpha}{2} - (2 - \frac{3}{2a})\frac{a\alpha}{4a-3} = 0, \\
&(\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) - c_1^{'} \le \frac{\alpha}{2} - (1 + \frac{1}{\theta_1})\frac{\theta_1\alpha}{2(1+\theta_1)} = 0.
\end{aligned}$$

   Thus, $(\frac{\alpha}{2} - \frac{1}{\theta_1}c_1^{'}) \le c_1^{'} < [\frac{\alpha}{2} - (1 - \frac{3}{2a}c_1^{'})] < (\frac{\alpha}{2} + \frac{c_1^{'}}{2a})$. Therefore, only the two regions corresponding to Types I and II equilibria have positive lengths.

3. If $\frac{a\alpha}{4a-3} \le c_1^{'} < \frac{\alpha}{2 - \frac{1}{a(1-\theta_1)+\theta_1}}$

   We have $[\frac{\alpha}{2} - (1 - \frac{3}{2a})c_1^{'}] \le c_1^{'} < (\frac{\alpha}{2} + \frac{c_1^{'}}{2a})$. Only the region corresponding to the Type I equilibrium has positive length.

This completes the proof. ∎

Note that in both Type I and Type II equilibria, all the flexible green consumers buy either the green variant or nothing. In the Type III equilibrium, some flexible green consumers switch to the regular variant. As a consequence, we have the following corollary.

Corollary 4.3.6 *When the competitor's cost disadvantage is marginal (c_2 is in Region III of Figure 4.4), Firm 1 does not have the power to retain all the green consumers. Some low-end flexible green consumers switch to the regular variant. When the competitor's cost disadvantage increases (c_2 is in either Region I or Region II), Firm 1 obtains this power.*

4.3.4 Presence of a Competitor When All Green Consumers Are Dedicated

When all green consumers are dedicated, they purchase either the green variant or nothing. Thus, the demand of the green variant is unaffected by the price of the regular variant. Accordingly, only the price of the green variant determines its demand. Recall in Section 4.2.2, we list the aggregate market demand under various scenarios. Due to the absence of the flexible green consumers, the aggregate market demand now has only two different forms when both regular and green variants are available.

(i) $p_g \leq p_r$

Under this scenario, all consumers purchase either the green variant or nothing. The aggregate market demand is as follows:

$$Q_r = 0, \quad Q_g = (\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

(ii) $p_g > p_r$

Under this scenario, the regular consumers purchase either the regular variant or nothing. Meanwhile, the green consumers purchase either the green variant or nothing.

The aggregate market demand is as follows:

$$Q_r = (\alpha - p_r)\theta_1, \quad Q_g = (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

Since our goal is to avoid exploring all theoretical possibilities and, instead, focus on practically-relevant scenarios in which the two firms enjoy healthy competition, we impose the following assumption: The two competing firms have similar production economies. In other words, both firms' variable production costs are of the same magnitude, both before and after Firm 1's implementation of the symbiotic system. Specifically, we assume that $2c_1 - \alpha < c_2 < \frac{\alpha + c_1}{2}$ and $2c_1^{'} - \alpha < c_2 < \frac{\alpha + c_1^{'}}{2}$. This assumption is reasonable in practice, since a significant difference in the production costs of the two firms will typically result in the weaker firm being driven out of the market.

Next, we first describe Model CR, in which Firm 1 faces a competitor (Firm 2) that only produces the regular variant of the product. The case in which the competitor produces the green variant (Model CG) is analyzed later.

No Symbiotic System For Firm 1 Under Model CR

In the absence of the symbiotic system, only the regular variant is available in the market. Thus, both firms compete only in the regular market, where all consumers have the same distribution of valuation. The following result states the equilibrium solution. Since the proof is straightforward, we avoid providing it.

Theorem 4.3.7 *In Model CR, if Firm 1 does not implement the symbiotic system, the equilibrium production quantities of the two firms are:* $q_1^* = \frac{\alpha - 2c_1 + c_2}{4b_1 + 3}$ *(Firm 1),* $q_2^* = \frac{(2b_1+1)\alpha + c_1 - 2(b_1+1)c_2}{4b_1+3}$ *(Firm 2).*

Symbiotic System For Firm 1 Under Model CR

If Firm 1 implements the symbiotic system, then it has the option of labeling its product as the green variant. The two choices for Firm 1 are (1) Strategy g: provide only the green variant or (2) Strategy rg: provide both the regular and green variants. Let π_1^g (resp., π_1^{rg}) denote Firm 1's profit under Strategy g (resp., Strategy rg). There are two questions: in equilibrium, (i) which strategy will Firm 1 choose? and (ii) what are the production quantities of each variant for each firm? The following result answers these questions.

Theorem 4.3.8 *In Model CR, if Firm 1 implements the symbiotic system, it will provide both regular and green variants. The equilibrium production quantities are:* $q_{1,r}^* = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)}{3}$, $q_{2,r}^* = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{3}$, *and* $q_{1,g}^* = \frac{(1-\theta_1)(a\alpha - c_1^{'})}{2a}$.

Proof: Recall that the aggregate market demand has two different forms under two scenarios. Since Firm 2 only produces the regular variant, the condition for Firm 2 to produce positive amount in equilibrium is $p_g > p_r$. Thus, we only consider this scenario in the following analysis. We consider Firm 1's two choices separately.

- **Strategy** rg

Firm 1 provides both regular and green variants, while Firm 2 provides the regular variant.

When $p_r < p_g$, we have $Q_r = (\alpha - p_r)\theta_1, Q_g = (\alpha - \frac{p_g}{a})(1 - \theta_1)$.

Firm 1's profit-maximization problem is as follows:

$$\begin{aligned}\max_{q_{1r}, p_g} \quad \pi_1 &= q_{1r}(p_r - c_1^{'}) + Q_g(p_g - c_1^{'}) \\ &= q_{1r}(\alpha - \frac{q_{1r} + q_{2r}}{\theta_1} - c_1^{'}) + (\alpha - \frac{p_g}{a})(1 - \theta_1)(p_g - c_1^{'}).\end{aligned}$$

Thus ,we have

$$\frac{\partial \pi_1}{\partial q_{1r}} = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_1^{'} - \frac{q_{1r}}{\theta_1} = 0 \tag{4.9}$$

$$\frac{\partial \pi_1}{\partial p_g} = (\alpha - \frac{p_g}{a})(1-\theta_1) - \frac{(1-\theta_1)(p_g - c_1^{'})}{a} = 0. \tag{4.10}$$

Firm 2 faces the following problem:

$$\max_{q_{2r}} \quad \pi_2 = (p_r - c_2)q_{2r} = (\alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2)q_{2r}.$$

We can obtain the first order condition:

$$\frac{\partial \pi_2}{\partial q_{2r}} = \alpha - \frac{q_{1r}+q_{2r}}{\theta_1} - c_2 - \frac{q_{2r}}{\theta_1} = 0 \tag{4.11}$$

By solving equations 4.9, 4.10, and 4.11 simultaneously, we obtain the following equilibrium:

$$q_{1,r}^* = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)}{3}, q_{2,r}^* = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{3}, q_{1,g}^* = \frac{(1-\theta_1)(a\alpha - c_1^{'})}{2a},$$

$$p_r^* = \frac{\alpha + c_1^{'} + c_2}{3}, p_g^* = \frac{a\alpha + c_1^{'}}{2}.$$

To check the validity of this solution, we first examine the difference between the prices of the two variants. We have $p_g^* - p_r^* = \frac{(3a-2)\alpha + c_1^{'} - 2c_2}{6} \geq \frac{\alpha + c_1^{'} - 2c_2}{6}$. Since $c_2 < \frac{\alpha + c_1^{'}}{2}$, we have $p_r^* < p_g^*$.

We then check whether all the production quantities are positive. Since $c_2 > 2c_1^{'} - \alpha$, we have $q_{1,r}^* > 0$. Since $c_2 < \frac{\alpha + c_1^{'}}{2}$, we have $q_{2,r}^* > 0$. Since $c_1^{'} < \alpha$ and $a \geq 1$, we have $q_{1,g}^* > 0$. Thus, the solution above is valid. Firm 1's profit under Strategy rg is $\pi_1^{rg} = \frac{(1-\theta_1)(a\alpha - c_1^{'})^2}{4a} + \frac{\theta_1(\alpha - 2c_1^{'} + c_2)^2}{9}$.

• **Strategy** g

Then Firm 1 only provides the green variant, while Firm 2 provides the regular variant. According to Assumption 3, Firm 2 produces positive amount of product. Thus, the price of the regular variant should be less than that of the green variant. Therefore, Firm 1 can only obtain profit from the green consumers. Note that, Firm 1's maximum profit from the green consumers is $\frac{(1-\theta_1)(a\alpha-c_1^{'})^2}{4a}$. Thus, Firm 1's profit under Strategy g, $\pi_1^g \leq \frac{(1-\theta_1)(a\alpha-c_1^{'})^2}{4a} < \pi_1^{rg}$.

The result follows. ■

Next, we consider the case in which a competitor has the ability to produce the green variant.

No Symbiotic System For Firm 1 Under Model CG

Under Model CG, we consider a competitor that also has the ability to produce the green variant. If Firm 1 does not implement the symbiotic system, then only Firm 2 has the option to label its product as the green variant. Firm 2 has two strategies, g and rg. The following result states the equilibrium.

Theorem 4.3.9 *In Model CG, if Firm 1 does not implement the symbiotic system, then Firm 2 will provide both regular and green variants. The equilibrium production quantities are:* $q_{1,r}^* = \frac{\theta_1(\alpha-2c_1+c_2)}{3+4b_1\theta_1}$, $q_{2,r}^* = \frac{\theta_1[(2b_1\theta_1+1)\alpha+c_1-2(b_1\theta_1+1)c_2]}{3+4b_1\theta_1}$, *and* $q_{2,g}^* = \frac{(1-\theta_1)(a\alpha-c_2)}{2a}$.

Proof: Since Firm 2 can provide either only the green variant or both the regular and the green variants, there are two possible scenarios.

1. Firm 2 chooses Strategy rg**:**

Firm 1 provides the regular variant and Firm 2 provides both regular and green variants.

By using an approach similar to that in the proof of Theorem 4.3.8, we have the following equilibrium:

$$q_{1,r}^* = \frac{\theta_1(\alpha - 2c_1 + c_2)}{3 + 4b_1\theta_1}, q_{2,r}^* = \frac{\theta_1[(2b_1\theta_1 + 1)\alpha + c_1 - 2(b_1\theta_1 + 1)c_2]}{3 + 4b_1\theta_1},$$

$$q_{2,g}^* = \frac{(1-\theta_1)(a\alpha - c_2)}{2a}, p_r^* = \frac{(2b_1\theta_1 + 1)\alpha + c_1 + (2b_1\theta_1 + 1)c_2}{3 + 4b_1\theta_1}, p_g^* = \frac{a\alpha + c_2}{2}.$$

To check the validity of this solution, we first examine the difference between the prices of the two variants. We have $p_g^* - p_r^* = \frac{[a(3+4b_1\theta_1)-(2+4b_1\theta_1)]\alpha - 2c_1 + c_2}{2(3+4b_1\theta_1)}$. Since $c_2 > 2c_1 - \alpha$, we have $[a(3 + 4b_1\theta_1) - (2 + 4b_1\theta_1)]\alpha - 2c_1 + c_2 > (a - 1)(3 + 4b_1\theta_1)\alpha \geq 0$. Thus, $p_g^* - p_r^* > 0$. Since $c_2 > 2c_1 - \alpha$, we have $q_{1,r}^* > 0$. Also, $c_2 < \frac{\alpha + c_1}{2}$ implies $[(2b_1\theta_1 + 1)\alpha + c_1 - 2(b_1\theta_1 + 1)c_2] > b_1\theta_1(\alpha - c_1) \geq 0$. Thus, we have $q_{2,r}^* > 0$. Since $c_2 < \alpha$ and $a \geq 1$, we have $q_{2,g}^* > 0$. Thus, the solution above is valid.

Firm 2's profit under Strategy rg is $\pi_2^{rg} = \frac{(1-\theta_1)(a\alpha - c_2)^2}{4a} + \frac{\theta_1[(2b_1\theta_1+1)\alpha + c_1 - 2(b_1\theta_1+1)c_2]^2}{(3+4b_1\theta_1)^2}$.

2. Firm 2 chooses Strategy g:

Then, Firm 1 provides the regular variant, while Firm 2 provides the green variant.

Since Firm 1 produces a positive amount (Assumption 3 above), the price of the regular variant is less than that of the green variant. Therefore, Firm 2 derives its entire profit from the green consumers; the maximum value of this profit is $\frac{(1-\theta_1)(a\alpha - c_2)^2}{4a}$. Thus, Firm 2's profit under Strategy g, $\pi_2^g \leq \frac{(1-\theta_1)(a\alpha - c_2)^2}{4a} < \pi_2^{rg}$.

The result follows. ∎

Symbiotic System For Firm 1 Under Model CG

Under this setting, both firms are capable of offering both variants. Each firm has two strategies. We denote $\pi_i^{j,k}$ as Firm i's profit if Firm 1 chooses Strategy j and Firm 2 chooses Strategy k; $i \in \{1, 2\}$, $j, k \in \{g, rg\}$. The following result shows that both firms provide both regular and green variants in equilibrium and derives the corresponding production quantities. The payoffs of the two firms under the four scenarios are listed in Table 4.5.

Table 4.5. The Two Firms' Payoffs when Both can Provide Regular and Green Variants.

		Firm 2's Strategy		
		Only Green		*Regular & Green*
Firm 1's	*Only Green*	$(\pi_1^{g,g}, \pi_2^{g,g})$	$\rightarrow$	$(\pi_1^{g,rg}, \pi_2^{g,rg})$
		$\downarrow$		$\downarrow$
Strategy	*Regular & Green*	$(\pi_1^{rg,g}, \pi_2^{rg,g})$	$\rightarrow$	$(\pi_1^{rg,rg}, \pi_2^{rg,rg})$

Theorem 4.3.10 *In Model CG, if Firm 1 implements the symbiotic system, then in equilibrium, both firms provide both the regular and the green variants. The equilibrium production quantities are:*

$$q_1^{r*} = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)}{3}, \quad q_1^{g*} = \frac{(1-\theta_1)(a\alpha - 2c_1^{'} + c_2)}{3a},$$
$$q_2^{r*} = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{3}, \quad q_2^{g*} = \frac{(1-\theta_1)(a\alpha + c_1^{'} - 2c_2)}{3a}.$$

Proof: Since each firm has two strategies, there are four possible combinations. We derive the payoffs of both firms under each combination in the following analysis.

1. Combination (g, g): If both firms only provide the green variant

   All consumers choose to buy either a green variant or nothing. The aggregate market demand is as follows:

$$q_{1,g} + q_{2,g} = Q_g = (\alpha - p_g)\theta_1 + (\alpha - \frac{p_g}{a})(1 - \theta_1).$$

Thus, $p_g = \frac{\alpha - q_{1,g} - q_{2,g}}{\theta_1 + \frac{1-\theta_1}{a}}$. The two firms face the following profit-maximization problems:

$$\begin{aligned}
\max_{q_{1,g}} \quad \pi_1 &= (p_g - c_1^{'})q_{1,g}, \\
\max_{q_{2,g}} \quad \pi_2 &= (p_g - c_2)q_{2,g}.
\end{aligned}$$

By using the method similar to that used in the proof of Theorem 4.3.8, we obtain the equilibrium results as follows:

$$p_g^* = \frac{1}{3}[c_1^{'} + c_2 + \frac{a\alpha}{1 + (a-1)\theta_1}],$$

$$q_{1,g}^* = \frac{a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1}{3a}, q_{2,g}^* = \frac{a\alpha + c_1^{'} - 2c_2 + (a-1)(c_1^{'} - 2c_2)\theta_1}{3a},$$

$$\pi_1^* = \frac{[a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1]^2}{9a[1 + (a-1)\theta_1},$$

$$\pi_2^* = \frac{[a\alpha + c_1^{'} - 2c_2 + (a-1)(c_1^{'} - 2c_2)\theta_1]^2}{9a[1 + (a-1)\theta_1}.$$

We check whether the production quantities derived are positive.

- If $(2c_1^{'} - c_2) \geq 0$, then $[a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1]$ reaches its minimum when θ_1 reaches its upper bound 1. Thus, $a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1 \geq a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2) = a(\alpha - 2c_1^{'} + c_2)$. Since $c_2 > 2c_1^{'} - \alpha$, we have $a(\alpha - 2c_1^{'} + c_2) > 0$. Thus, $q_{1,g}^* > 0$.

- If $(2c_1^{'} - c_2) < 0$, then $[a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1]$ reaches its minimum when θ_1 reaches its lower bound 0. Thus, $a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1 \geq a\alpha - 2c_1^{'} + c_2 \geq (\alpha - 2c_1^{'} + c_2) > 0$. Thus, $q_{1,g}^* > 0$.

Similarly, we can show $q_{2,g}^* > 0$. Thus, the solution above is valid. These two firms' profits are as follows.

$$\pi_1^{g,g} = \frac{[a\alpha - 2c_1^{'} + c_2 - (a-1)(2c_1^{'} - c_2)\theta_1]^2}{9a[1+(a-1)\theta_1]},$$

$$\pi_2^{g,g} = \frac{[a\alpha + c_1^{'} - 2c_2 + (a-1)(c_1^{'} - 2c_2)\theta_1]^2}{9a[1+(a-1)\theta_1]}.$$

2. Combination (rg, rg): If both firms provide both regular and green variants

Under this scenario, both firms compete in both the regular and the green markets. We obtain the equilibrium results as follows:

$$p_r^* = \frac{\alpha + c_1^{'} + c_2}{3}, p_g^* = \frac{a\alpha + c_1^{'} + c_2}{3}, q_{1,r}^* = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)}{3},$$

$$q_{2,r}^* = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)}{3}, q_{1,g}^* = \frac{(1-\theta_1)(a\alpha - 2c_1^{'} + c_2)}{3a}, q_{2,g}^* = \frac{(1-\theta_1)(a\alpha + c_1^{'} - 2c_2)}{3a}.$$

We have $p_g^* - p_r^* = \frac{(a-1)\alpha}{3} > 0$. We can show all four production quantities are positive. These two firms' profits are as follows.

$$\pi_1^{rg,rg} = \frac{\theta_1(\alpha - 2c_1^{'} + c_2)^2}{9} + \frac{(1-\theta_1)(a\alpha - 2c_1^{'} + c_2)^2}{9a},$$

$$\pi_2^{rg,rg} = \frac{\theta_1(\alpha + c_1^{'} - 2c_2)^2}{9} + \frac{(1-\theta_1)(a\alpha + c_1^{'} - 2c_2)^2}{9a}.$$

If we compare the profits under this setting with those under (g, g), we have

$$\pi_1^{rg,rg} - \pi_1^{g,g} = \frac{\theta_1(1-\theta_1)(a-1)^2\alpha^2}{9[1+(a-1)\theta_1]} \geq 0,$$

$$\pi_2^{rg,rg} - \pi_2^{g,g} = \frac{\theta_1(1-\theta_1)(a-1)^2\alpha^2}{9[1+(a-1)\theta_1]} \geq 0.$$

3. Combination (g, rg): If Firm 1 only provides the green variant, Firm 2 provides both regular and green variants

We now compare these two firms' profits under this setting and those under the setting (rg, rg). First, both firms obtain the same amount of profits from the green market under both settings. Second, Firm 1 produces the regular variant under the setting (rg, rg) but not under the setting (g, rg). Thus, in the regular market, Firm 1 obtains positive profit under the setting (rg, rg) but 0 under the setting (g, rg). Third, in the regular market, Firm 2 competes with Firm 1 under the setting (rg, rg) but is the exclusive supplier under the setting (g, rg). Thus, Firm 2 obtains more profit in the regular market under the setting (rg, rg) than under the setting (g, rg). Therefore, we have the following results:

$$\pi_1^{rg,rg} > \pi_1^{g,rg}, \quad \pi_2^{g,rg} > \pi_2^{rg,rg}.$$

4. Combination (rg, g): If Firm 1 provides both regular and green variants, Firm 2 only provides the green variant

   This scenario is similar to Combination (g, rg). If we compare these two firms' profits under this setting and under the setting (rg, rg), we have the following results:

$$\pi_1^{rg,g} > \pi_1^{rg,rg}, \quad \pi_2^{rg,rg} > \pi_2^{rg,g}.$$

Now we derive the Nash equilibrium. Since we have $\pi_1^{rg,g} > \pi_1^{rg,rg} > \pi_1^{g,g}$ and $\pi_1^{rg,rg} > \pi_1^{g,rg}$, Firm 1's dominating strategy is rg no matter which strategy Firm 2 chooses. Similarly, we have $\pi_2^{rg,rg} > \pi_2^{rg,g}$ and $\pi_2^{g,rg} > \pi_2^{rg,rg} > \pi_2^{g,g}$. Thus, Firm 2's dominating strategy is rg as well. Therefore, (rg, rg) is the only Nash equilibrium. Both Firms provide both the regular and the green variants. This completes the proof. ∎

In the next section, we use the obtained results to derive some useful managerial insights.

4.4 Understanding the Willingness to Implement the Symbiotic System

In the previous section, we derived Firm 1's optimal (under monopoly) and equilibrium (under competition) decisions, under various settings. These results enable us to assess the firm's profits both before and after the implementation of a symbiotic system. We interpret the difference between these two profits as the firm's "willingness" to implement the system. Note that this difference may not always be positive. Accordingly, our interest is in understanding the forces that influence willingness and the impact on their relative strengths with respect to changes in operational parameters, consumer characteristics, and competition. Section 4.4.1 (resp., Section 4.4.2) discusses the situation under Model M (resp., Model CR). We also briefly comment on the main differences under Model CG. Next, in Section 4.4.3, we compare the firm's willingness under monopoly with that under competition. Our focus is on analyzing the impact of the nature of competition (competitors offering only the regular variants vs. both variants) and on identifying scenarios where competition improves the firm's willingness to implement. Finally, Section 4.4.4 discusses the impact of the implementation of a symbiotic system on consumer welfare. Here, our aim is to identify situations in which the firm's willingness to implement is positive and consumers (as a whole) benefit from the implementation as well.

For simplicity, when considering competition, we assume that green consumers are dedicated. The case when green consumers are flexible can also be analyzed, but is technically more cumbersome. To calculate the firm's profits before and after implementation, we need to consider the decisions for both Product P and Product S. Therefore, we introduce the corresponding subscripts (p and s) to the notation defined for the single-product models in earlier sections. Let π_1^* (resp., $\pi_1^{'*}$) denote Firm 1's maximum total profit before (resp., after) implementing the symbiotic system.

4.4.1 The Willingness in a Monopoly

Recall from Section 4.3.1 the optimal decisions under monopoly, obtained under the assumptions $\max\{c_{1,p}, c_{1,p}^{'}\} < \alpha_p$ and $\max\{c_{1,s}, c_{1,s}^{'}\} < \alpha_s$. Before implementing the symbiotic system, Firm 1's total profit from products P and S is:

$$\pi_1^* = \frac{(\alpha_p - c_{1,p})^2}{4(1+b_1)} + \frac{(\alpha_s - c_{1,s})^2}{4(1+b_1)}.$$

The total profit after implementation is:

$$\begin{aligned}\pi_1^{'*} = \quad & \frac{\theta_{1,p}(\alpha_p - c_{1,p}^{'})^2}{4} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p}^{'})^2}{4a_p} + \frac{\theta_{1,s}(\alpha_s - c_{1,s}^{'})^2}{4} + \\ & \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s}^{'})^2}{4a_s} - K.\end{aligned}$$

To capture the firm's willingness to implement, let $\Delta_M = \pi_1^{'*} - \pi_1^*$. To better understand the impact of parametric changes, it is convenient to partition the expression for Δ_M into four terms:

$$\begin{aligned}\Delta_M \quad = \quad & \underbrace{\frac{(1-\theta_{1,p})(a_p - 1)(a_p\alpha_p^2 - c_{1,p}^{'2})}{4a_p} + \frac{(1-\theta_{1,s})(a_s - 1)(a_s\alpha_s^2 - c_{1,s}^{'2})}{4a_s}}_{T_1^M:\text{ The gain from exploiting the green market segment}} \\ & + \underbrace{\frac{(c_{1,p} - c_{1,p}^{'})(2\alpha_p - c_{1,p} - c_{1,p}^{'})}{4} + \frac{(c_{1,s} - c_{1,s}^{'})(2\alpha_s - c_{1,s} - c_{1,s}^{'})}{4}}_{T_2^M:\text{ The gain/loss from the changes in the variable production costs}} \\ & + \underbrace{\frac{b_1(\alpha_p - c_{1,p})^2}{4(1+b_1)} + \frac{b_1(\alpha_s - c_{1,s})^2}{4(1+b_1)}}_{T_3^M:\text{ The gain from better usage of the waste}} - \underbrace{K}_{T_4^M:\text{ The fixed cost}}.\end{aligned}$$

These four terms can be further categorized into two types: Demand-side and Cost-side.

- Demand-side Influence:

$T_1^M = \frac{(1-\theta_{1,p})(a_p-1)(a_p\alpha_p^2-c_{1,p}^{'2})}{4a_p} + \frac{(1-\theta_{1,s})(a_s-1)(a_s\alpha_s^2-c_{1,s}^{'2})}{4a_s}$

This represents the additional revenue for Firm 1 from providing the green variants of the two products (instead of the regular variants) to the green consumers. Since the green consumers have higher valuation for the green variants (relative to their regular counterparts), this term is always non-negative. If either (i) the green consumers do not exist (i.e., $\theta_{1,p} = 1$ and $\theta_{1,s} = 1$), or (ii) the green consumers value the green variants the same as the regular variants (i.e., $a_p = 1$; $a_s = 1$), then we have $T_1^M = 0$.

- Cost-side Influence:

1. $T_2^M = \frac{(c_{1,p}-c_{1,p}^{'})(2\alpha_p-c_{1,p}-c_{1,p}^{'})}{4} + \frac{(c_{1,s}-c_{1,s}^{'})(2\alpha_s-c_{1,s}-c_{1,s}^{'})}{4}$

   This represents the gain/loss from changes in the variable production costs. If the production costs of both products reduce after the implementation of the symbiotic system (i.e., $c_{1,p}^{'} \leq c_{1,p}$ and $c_{1,s}^{'} \leq c_{1,s}$), then this term represents a benefit for Firm 1. Otherwise, if both production costs increase, then the term reflects a loss. In general, T_2^M can be either a benefit or a loss. If the variable production costs of both products remain the same after implementation (i.e., $c_{1,p} = c_{1,p}^{'}$ and $c_{1,s} = c_{1,s}^{'}$), then we have $T_2^M = 0$.

2. $T_3^M = \frac{b_1(\alpha_p-c_{1,p})^2}{4(1+b_1)} + \frac{b_1(\alpha_s-c_{1,s})^2}{4(1+b_1)}$

   This represents the saving for Firm 1 in the waste treatment cost. Recall that one benefit of implementation is the simplification in the treatment process of the wastes of both the products. This term is always non-negative. If the waste treatment cost had a linear form before the implementation of the symbiotic system (i.e., $b_1 = 0$), then we have $T_3^M = 0$.

3. $T_4^M = K$

   This is the fixed cost incurred by Firm 1 in implementing the symbiotic system.

Our interest is in gaining insights on the conditions under which $\Delta_M \geq 0$, since this would indicate the firm's willingness to implement the system. The classification above implies the following remark.

Remark 1 In a monopoly, if the combined benefit of (i) exploiting the green market segment (T_1^M), (ii) better usage of the waste (T_3^M), and (iii) lowering the variable production cost (T_2^M, if $T_2^M > 0$) is more than the sum of the (1) higher variable production costs ($-T_2^M$, if $T_2^M < 0$) and (2) fixed cost (T_4^M), then the firm's willingness (Δ_M) to implement the symbiotic system is nonnegative. ■

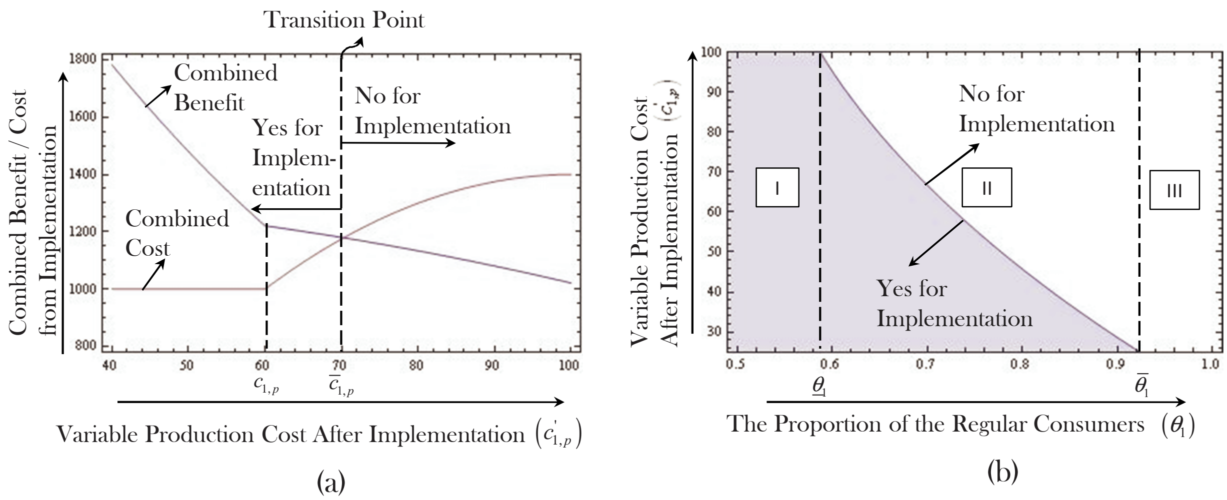


Figure 4.5. Willingness in a Monopoly: Influence on the Decision to Implement of the Change in the (i) Variable Production Cost after Implementation ($c'_{1,p}$), and (ii) Proportion of Regular Consumers of Product P ($\theta_{1,p}$).

Figure 4.5(a) illustrates Firm 1's combined benefit and combined cost from implementation with a change in the variable production cost after implementation ($c'_{1,p}$), while keeping all other parameters fixed. To allow us to focus on one product (say Product P), we assume $c_{1,s} = c'_{1,s}$ for simplicity. When $c'_{1,p}$ is less than the value of the variable production cost $c_{1,p}$ before implementation, we have $T_2^M \geq 0$. Thus, when $c'_{1,p} \leq c_{1,p}$, the combined benefit consists of three terms: T_1^M, T_2^M, and T_3^M, while the combined cost is T_4^M. When $c'_{1,p} > c_{1,p}$, the term T_2^M is negative. Thus, the combined benefit now consists of only two terms: T_1^M and T_3^M, while the combined cost has two terms: $-T_2^M$ and T_4^M. As shown in the figure, the curve representing the combined benefit (resp., combined cost) of implementing the system decreases (resp., increases) with an increase in $c'_{1,p}$. The willingness measure Δ_M changes sign at the threshold production cost $\bar{c}_{1,p}$. As long as $c'_{1,p}$ is smaller than this threshold, the combined benefit exceeds the combined cost.

In Figure 4.5(b), we illustrate the impact of the proportion of regular consumers of Product P ($\theta_{1,p}$) on the firm's decision to implement the system. Note that (i) $c'_{1,p}$ is between 0 and α_p and (ii) the value of Δ_M decreases with an increase in $c'_{1,p}$.

First, consider the (worst-case) situation when $c'_{1,p}$ is at its upper bound α_p. In this case, the gain from exploiting the green market is positive (i.e., $T_1^M > 0$), while the firm incurs a loss from changes in the variable product costs (i.e., $T_2^M < 0$). Also, recall that the firm gains from the better usage of the wastes of the two products (i.e., $T_3^M \geq 0$). In Region I of Figure 4.5(b), where $\theta_{1,p}$ is less than a threshold $\underline{\theta}_{1,p}$, the combined benefit of T_1^M and T_3^M dominates the combined loss of $-T_2^M$ and T_4^M. Consequently, we have $\Delta_M > 0$. Since Δ_M increases with a decrease in $c'_{1,p}$, we continue to have $\Delta_M > 0$ as $c'_{1,p}$ reduces from α_p to 0.

Next, consider the (best-case) situation when $c'_{1,p}$ is at its lower bound 0. Here, the firm benefits from changes in the variable product costs (i.e., $T_2^M > 0$). Thus, the combined benefit consists of three terms T_1^M, T_2^M, and T_3^M. However, the contribution of T_1^M decreases with an increase in $\theta_{1,p}$. Therefore, in Region III of Figure 4.5(b), where $\theta_{1,p}$ is greater than a threshold $\overline{\theta}_{1,p}$, the combined cost T_4^M dominates the combined benefit of T_1^M, T_2^M, and T_3^M. Consequently, we have $\Delta_M < 0$. Again, as Δ_M decreases with an increase in $c'_{1,p}$, we continue to have $\Delta_M < 0$ as $c'_{1,p}$ increases from 0 to α_p.

In Region II, when the value of $\theta_{1,p}$ is between the two thresholds $\underline{\theta}_{1,p}$ and $\overline{\theta}_{1,p}$, there is a healthier tradeoff between the combined cost and the combined benefit. For any value of $\theta_{1,p}$ in this region, there exists a threshold of $c'_{1,p}$ (represented by the curve) below which the decision for implementation is in the affirmative.

In the next section, we consider the case under Model CR.

4.4.2 The Willingness under Competition for the Regular Variants

Recall the optimal decisions under competition (Section 4.3.4), obtained under the assumptions $\max\{2c_{1,p} - \alpha_p, 2c'_{1,p} - \alpha_p, 0\} < c_{2,p} < \min\{\frac{\alpha_p + c_{1,p}}{2}, \frac{\alpha_p + c'_{1,p}}{2}\}$ and $\max\{2c_{1,s} - \alpha_s, 2c'_{1,s} - \alpha_s, 0\} < c_{2,s} < \min\{\frac{\alpha_s + c_{1,s}}{2}, \frac{\alpha_s + c'_{1,s}}{2}\}$. Firm 1's total profit from the two products before implementation is:

$$\pi_1^* = \frac{(b_1+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1+3)^2} + \frac{(b_1+1)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{(4b_1+3)^2}.$$

After implementation, the total profit is:

$$\begin{aligned}
\pi_1^{'*} &= \frac{\theta_{1,p}(\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p}^{'})^2}{4a_p} \\
&\quad + \frac{\theta_{1,s}(\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s}^{'})^2}{4a_s} - K.
\end{aligned}$$

Let $\Delta_{CR} = \pi_1^{'*} - \pi_1^*$. We partition Δ_{CR} into five terms that can be conveniently interpreted:

$$\begin{aligned}
\Delta_{CR} &= \pi_1^{'*} - \pi_1^* \\
&= \underbrace{(1-\theta_{1,p})[\frac{(a_p\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9a_p} - \frac{(\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9}] + (1-\theta_{1,s})[\frac{(a_s\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9a_s} - \frac{(\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9}]}_{T_1^R\text{: The gain from exploring the green market segment}} \\
&\quad \underbrace{+(1-\theta_{1,p})[\frac{(a_p\alpha_p - c_{1,p}^{'})^2}{4a_p} - \frac{(a_p\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9a_p}] + (1-\theta_{1,s})[\frac{(a_s\alpha_s - c_{1,s}^{'})^2}{4a_s} - \frac{(a_s\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9a_s}]}_{T_2^R\text{: The benefit of being the exclusive supplier of the green market segment}} \\
&\quad + \underbrace{\frac{4(c_{1,p} - c_{1,p}^{'})(\alpha_p - c_{1,p} - c_{1,p}^{'} + c_{2,p})}{9} + \frac{4(c_{1,s} - c_{1,s}^{'})(\alpha_s - c_{1,s} - c_{1,s}^{'} + c_{2,s})}{9}}_{T_3^R\text{: The gain/loss from the changes in the variable production costs}} \\
&\quad + \underbrace{\frac{b_1(16b_1+15)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9(4b_1+3)^2} + \frac{b_1(16b_1+15)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9(4b_1+3)^2}}_{T_4^R\text{: The gain from better usage of the waste}} \\
&\quad - \underbrace{K}_{T_5^R\text{: The fixed cost}}.
\end{aligned}$$

T_1^R, T_3^R, T_4^R and T_5^R have similar corresponding terms (T_1^M, T_2^M, T_3^M and T_4^M, respectively) in the expression of Δ_M in Section 4.4.1. The only term unique to Δ_{CR} is T_2^R. We, therefore, discuss this expression here.

$$\begin{aligned}
T_2^R &= (1-\theta_{1,p})[\frac{(a_p\alpha_p - c_{1,p}^{'})^2}{4a_p} - \frac{(a_p\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9a_p}] \\
&\quad + (1-\theta_{1,s})[\frac{(a_s\alpha_s - c_{1,s}^{'})^2}{4a_s} - \frac{(a_s\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9a_s}]
\end{aligned}$$

This term represents the additional revenue of Firm 1 for being the *exclusive supplier to the green consumers*. It can be shown that $T_2^R \geq 0$, with equality holding only when green consumers do not exist (i.e., $\theta_{1,p} = 1$ and $\theta_{1,s} = 1$). Under competition for the regular variants, the demand-side impact is represented by T_1^R and T_2^R while the cost-side influence is reflected in T_3^R, T_4^R, and T_5^R. This implies our next remark. The subsequent discussion offers insights on the willingness to implement.

Remark 2 Under competition for the regular variants of the two products (Model CR), if the combined benefit of (i) exploiting the green consumers (T_1^R), (ii) being the exclusive supplier of the green market segment (T_2^R), (iii) better usage of the waste (T_4^R), and (iv) lowering the variable production cost (T_3^R, if $T_3^R > 0$) exceeds the sum of the (1) higher variable production cost ($-T_3^R$, if $T_3^R < 0$) and (2) fixed cost (T_5^R), then the firm's willingness (Δ_{CR}) to implement the symbiotic system is nonnegative. ■

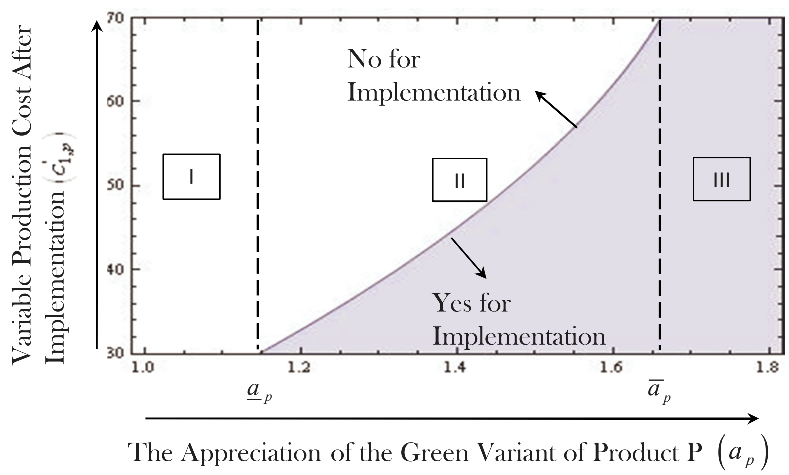


Figure 4.6. Willingness under Competition for Regular Variants: Influence on the Decision to Implement of the Change in the Appreciation of the Green Variant of Product P (a_p).

Figure 4.6 illustrates the impact of the appreciation a_p of the green variant of Product P on the firm's decision to implement the system. Again, note that (i) $c'_{1,p}$ is between $c'_{lb} =$

$\max\{2c_{2,p} - \alpha_p, 0\}$ and $c'_{ub} = \frac{\alpha_p + c_{2,p}}{2}$ and (ii) the value of Δ_{CR} decreases with an increase in $c'_{1,p}$. When $c'_{1,p}$ is at its lower bound c'_{lb}, the firm benefits from changes in the variable product costs (i.e., $T_3^R > 0$). Thus, the combined benefit consists of four terms T_1^R, T_2^R, T_3^R, and T_4^R. However, the contributions of T_1^R and T_2^R are marginal when a_p is close to its lower bound 1. Therefore, in Region I of Figure 4.6, where a_p is less than a threshold $\underline{a}_p$, the cost T_5^R dominates the combined benefit of T_1^R, T_2^R, T_3^R, and T_4^R. Consequently, we have $\Delta_{CR} < 0$. Since Δ_{CR} decreases with an increase in $c'_{1,p}$, we continue to have $\Delta_{CR} < 0$ as $c'_{1,p}$ increases from c'_{lb} to c'_{ub}. Now consider the situation when $c'_{1,p}$ is at its upper bound c'_{ub}. The firm incurs a loss from changes in the variable product costs (i.e., $T_3^R < 0$). However, the contributions of T_1^R and T_2^R increase with an increase of a_p. In Region III of Figure 4.6, where a_p is more than a threshold $\overline{a}_p$, the combined benefit of T_1^R, T_2^R, and T_4^R dominates the combined loss of $-T_3^R$ and T_5^R. Consequently, we have $\Delta_{CR} > 0$ and (since $\Delta_{CR} > 0$ increases as $c'_{1,p}$ reduces) this continues to hold as $c'_{1,p}$ reduces from c'_{ub} to c'_{lb}. As in Figure 4.6, in the intermediate region (Region II), where the value of a_p is between the two thresholds $\underline{a}_p$ and $\overline{a}_p$, there exists a threshold of $c'_{1,p}$ (represented by the curve) below which the willingness to implement the symbiotic system is positive.

When competitors can produce both regular and green variants (Model CG), the behavior of the the firm's willingness to implement the symbiotic system can be analyzed in a similar manner. We, therefore, avoid providing a detailed description here and briefly list the main change in the expression of the willingness Δ_{CG} under Model CG over Δ_{CR}.

In the willingness under Model CG, there exists an additional term reflecting the access to the green market. Before implementation of the system, under Model CR, both Firm 1 and

Firm 2 provide the regular variant. Since the green variant is unavailable, Firm 1 provides the regular variant to the entire market which includes the dedicated green consumers. Under Model CG, Firm 2 produces the green variant. Since the dedicated green consumers only purchase the green variant, Firm 1 cannot access the green market before the implementation. After implementation, under Model CR, Firm 1 exclusively supplies the green variant to the green consumers. Under Model CG, Firm 1 gains access to the green market after implementation.

In the next section, our aim is to investigate how a change in the nature of competition affects the firm's willingness to implement the symbiotic system.

4.4.3 A Comparative Look at the Shift in Willingness under Competition

We first summarize Firm 1's willingness to implement the symbiotic system under Models M, CR, and CG, in the following table.

Table 4.6. Willingness to Implement the Symbiotic System Under Monopoly, Competition for the Regular Variant, and Competition for Both Regular and Green Variants.

Δ_M	$\frac{\theta_{1,p}(\alpha_p-c'_{1,p})^2}{4}+\frac{(1-\theta_{1,p})(a_p\alpha_p-c'_{1,p})^2}{4a_p}-\frac{(\alpha_p-c_{1,p})^2}{4(1+b_1)}+\frac{\theta_{1,s}(\alpha_s-c'_{1,s})^2}{4}+\frac{(1-\theta_{1,s})(a_s\alpha_s-c'_{1,s})^2}{4a_s}-\frac{(\alpha_s-c_{1,s})^2}{4(1+b_1)}-K$
Δ_{CR}	$\frac{\theta_{1,p}(\alpha_p-2c'_{1,p}+c_{2,p})^2}{9}+\frac{(1-\theta_{1,p})(a_p\alpha_p-c'_{1,p})^2}{4a_p}-\frac{(b_1+1)(\alpha_p-2c_{1,p}+c_{2,p})^2}{(4b_1+3)^2}+\frac{\theta_{1,s}(\alpha_s-2c'_{1,s}+c_{2,s})^2}{9}+\frac{(1-\theta_{1,s})(a_s\alpha_s-c'_{1,s})^2}{4a_s}-\frac{(b_1+1)(\alpha_s-2c_{1,s}+c_{2,s})^2}{(4b_1+3)^2}-K$
Δ_{CG}	$\frac{\theta_{1,p}(\alpha_p-2c'_{1,p}+c_{2,p})^2}{9}+\frac{(1-\theta_{1,p})(a_p\alpha_p-2c'_{1,p}+c_{2,p})^2}{9a_p}-\frac{\theta_{1,p}(b_1\theta_{1,p}+1)(\alpha_p-2c_{1,p}+c_{2,p})^2}{(4b_1\theta_{1,p}+3)^2}+\frac{\theta_{1,s}(\alpha_s-2c'_{1,s}+c_{2,s})^2}{9}+\frac{(1-\theta_{1,s})(a_s\alpha_s-2c'_{1,s}+c_{2,s})^2}{9a_s}-\frac{\theta_{1,s}(b_1\theta_{1,s}+1)(\alpha_s-2c_{1,s}+c_{2,s})^2}{(4b_1\theta_{1,s}+3)^2}-K$

In general, with the introduction of competition, Firm 1 produces less quantities of the two products and earns less profit. Competition also hurts Firm 1's market share. For example, if the competitor produces the green variant, then Firm 1 shares the green market after the implementation of the symbiotic system. Since one of the benefits of the implementation is to exploit the green market, competition reduces this benefit. Thus, an intuitive projection would be that Firm 1 would be less willing to implement the system under competition than under monopoly. A natural question arises: *Can the firm's willingness to implement the symbiotic system increase under competition, relative to that under a monopoly?* In Theorems 4.4.1, 4.4.3, and 4.4.5 and their corresponding corollaries, our effort is to (1) identify and interpret scenarios where competition results in an increase in willingness and (2) understand the relative impact of the nature of the competition.

Theorem 4.4.1 *If* $c'_{1,p} = c_{1,p}$, $c'_{1,s} = c_{1,s}$, $0 \leq \theta_{1,p} \leq \frac{1}{b_1+1}$, *and* $0 \leq \theta_{1,s} \leq \frac{1}{b_1+1}$, *then we have* $\Delta_{CR} \geq \Delta_M$.

Proof: If $c'_{1,p} = c_{1,p}$ and $c'_{1,s} = c_{1,s}$, then we have

$$\begin{aligned}
& \Delta_M - \Delta_{CR} \\
= \; & \frac{(\alpha_p - c_{1,p})^2}{4}(\theta_{1,p} - \frac{1}{b_1+1}) - \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9}[\theta_{1,p} - \frac{9(b_1+1)}{(4b_1+3)^2}] \\
& + \frac{(\alpha_s - c_{1,s})^2}{4}(\theta_{1,s} - \frac{1}{b_1+1}) - \frac{(\alpha_s - 2c'_{1,s} + c_{2,s})^2}{9}[\theta_{1,s} - \frac{9(b_1+1)}{(4b_1+3)^2}] \\
= \; & [\frac{(\alpha_p - c_{1,p})^2}{4} - \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9}](\theta_{1,p} - \frac{1}{b_1+1}) \\
& + [\frac{(\alpha_s - c_{1,s})^2}{4} - \frac{(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9}](\theta_{1,s} - \frac{1}{b_1+1}) \\
& - \frac{b_1(7b_1+6)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9(b_1+1)(4b_1+3)^2} - \frac{b_1(7b_1+6)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9(b_1+1)(4b_1+3)^2}
\end{aligned}$$

Since $\frac{(\alpha_p - c_{1,p})}{2} - \frac{(\alpha_p - 2c_{1,p} + c_{2,p})}{3} = \frac{(\alpha_p + c_{1,p} - 2c_{2,p})}{6} > 0$, we have

$$\frac{(\alpha_p - c_{1,p})^2}{4} - \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} > 0.$$

Similarly, we have

$$\frac{(\alpha_s - c_{1,s})^2}{4} - \frac{(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9} > 0.$$

Also, $\frac{b_1(7b_1+6)(\alpha_p-2c_{1,p}+c_{2,p})^2}{9(b_1+1)(4b_1+3)^2} \geq 0$, and $\frac{b_1(7b_1+6)(\alpha_s-2c_{1,s}+c_{2,s})^2}{9(b_1+1)(4b_1+3)^2} \geq 0$. Thus, if $\theta_{1,p} - \frac{1}{b_1+1} \leq 0$ and $\theta_{1,s} - \frac{1}{b_1+1} \leq 0$, then $\Delta_M \leq \Delta_{CR}$. ■

Corollary 4.4.2 *If Firm 1's variable production costs of both products remain the same after the implementation of the symbiotic system, and the proportions of green consumers in both products' markets are more than a threshold, then competing with firms who only produce regular variants encourages the firm to implement the symbiotic system.*

Theorem 4.4.3 *If $c'_{1,p} = c_{1,p}$, $c'_{1,s} = c_{1,s}$, and $b_1 = 0$, then*

(i) If $a_p = 1$ and $a_s = 1$, we have $\Delta_{CG} \geq \Delta_M$.

(ii) If $c_{2,p} \leq c_{1,p}$, $c_{2,s} \leq c_{1,s}$, $a_p \geq \frac{9}{5}$, and $a_s \geq \frac{9}{5}$, we have $\Delta_{CG} \leq \Delta_M$.

Proof: If $c'_{1,p} = c_{1,p}$, $c'_{1,s} = c_{1,s}$, and $b_1 = 0$, then

$$\begin{aligned} & \Delta_{CG} - \Delta_M \\ = \ & \frac{(1-\theta_{1,p})(a_p\alpha_p - 2c_{1,p} + c_{2,p})^2}{9a_p} + \frac{(1-\theta_{1,p})(\alpha_p - c_{1,p})^2}{4} - \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p})^2}{4a_p} \\ & + \frac{(1-\theta_{1,s})(a_s\alpha_s - 2c_{1,s} + c_{2,s})^2}{9a_s} + \frac{(1-\theta_{1,s})(\alpha_s - c_{1,s})^2}{4} - \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s})^2}{4a_s} \end{aligned}$$

(i) If $a_p = 1$ and $a_s = 1$, then

$$\Delta_{CG} - \Delta_M = \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,s})(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9} \geq 0.$$

ii) If $c_{2,p} \le c_{1,p}$ and $c_{2,s} \le c_{1,s}$, then

$$\begin{aligned}
& \Delta_{CG} - \Delta_M \\
\le\ & \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p})^2}{9a_p} + \frac{(1-\theta_{1,p})(\alpha_p - c_{1,p})^2}{4} - \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p})^2}{4a_p} \\
& + \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s})^2}{9a_s} + \frac{(1-\theta_{1,s})(\alpha_s - c_{1,s})^2}{4} - \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s})^2}{4a_s} \\
=\ & \frac{(1-\theta_{1,p})}{36a_p}[(-5a_p^2 + 9a_p)\alpha_p^2 - 8a_p\alpha_p c_{1,p} + (9a_p - 5)c_{1,p}^2] \\
& + \frac{(1-\theta_{1,s})}{36a_s}[(-5a_s^2 + 9a_s)\alpha_s^2 - 8a_s\alpha_s c_{1,s} + (9a_s - 5)c_{1,s}^2]
\end{aligned}$$

If $a_p \ge \frac{9}{5}$, then $(-5a_p^2 + 9a_p) \le 0$. Since $\alpha_p > c_{1,p}$, we have

$$\begin{aligned}
& \frac{(1-\theta_{1,p})}{36a_p}[(-5a_p^2 + 9a_p)\alpha_p^2 - 8a_p\alpha_p c_{1,p} + (9a_p - 5)c_{1,p}^2] \\
\le\ & \frac{(1-\theta_{1,p})}{36a_p}[(-5a_p^2 + 9a_p)c_{1,p}^2 - 8a_p c_{1,p}^2 + (9a_p - 5)c_{1,p}^2] \\
=\ & \frac{-5(1-\theta_{1,p})(a_p - 1)^2 c_{1,p}^2}{36a_p} \le 0.
\end{aligned}$$

Similarly, we have $\frac{(1-\theta_{1,s})}{36a_s}[(-5a_s^2 + 9a_s)\alpha_s^2 - 8a_s\alpha_s c_{1,s} + (9a_s - 5)c_{1,s}^2] \le 0$. Therefore, we have $\Delta_{CG} \le \Delta_M$.

■

Corollary 4.4.4 *If the implementation of the symbiotic system does not change the variable production costs of both products, and the savings on the waste treatment cost are negligible, then*

(1) Even if the dedicated green consumers' appreciation for the green variants is marginal, the firm is more willing (than in a monopoly) to implement the symbiotic system when competing with firms who produce green variants.

(2) *Even if the dedicated green consumers' appreciation for the green variants is higher than a threshold, competing with firms who produce green variants discourages the firm to implement the symbiotic system if the competitors have cost advantages over Firm 1 for both products.*

Theorem 4.4.5 *If* $b_1 = 0$, $c_{2,p} \leq c_{1,p} = c^{'}_{1,p}$, *and* $c_{2,s} \leq c_{1,s} = c^{'}_{1,s}$, *then* $\Delta_{CG} < \Delta_{CR}$.

Proof: If $b_1 = 0$, $c^{'}_{1,p} = c_{1,p}$, and $c^{'}_{1,s} = c_{1,s}$, then we have

$$\begin{aligned}
& \Delta_{CG} - \Delta_{CR} \\
= \; & (1-\theta_{1,p})[\frac{(a_p\alpha_p - 2c_{1,p} + c_{2,p})^2}{9a_p} - \frac{(a_p\alpha_p - c_{1,p})^2}{4a_p} + \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9}] \\
& +(1-\theta_{1,s})[\frac{(a_s\alpha_s - 2c_{1,s} + c_{2,s})^2}{9a_s} - \frac{(a_s\alpha_s - c_{1,s})^2}{4a_s} + \frac{(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9}]
\end{aligned}$$

Since $\frac{\partial \frac{(a_p\alpha_p - 2c_{1,p}+c_{2,p})^2}{9a_p}}{\partial a_p} = \frac{(a_p\alpha_p-2c_{1,p}+c_{2,p})(a_p\alpha_p+2c_{1,p}-c_{2,p})}{9a_p^2} > 0$, the quantity $\frac{(a_p\alpha_p-2c_{1,p}+c_{2,p})^2}{9a_p}$ increases with an increase in a_p. Thus, for $a_p \geq 1$, we have $\frac{(a_p\alpha_p-2c_{1,p}+c_{2,p})^2}{9a_p} \geq \frac{(\alpha_p-2c_{1,p}+c_{2,p})^2}{9}$. Therefore,

$$\begin{aligned}
& \frac{(a_p\alpha_p - 2c_{1,p} + c_{2,p})^2}{9a_p} - \frac{(a_p\alpha_p - c_{1,p})^2}{4a_p} + \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} \\
\leq \; & \frac{2(a_p\alpha_p - 2c_{1,p} + c_{2,p})^2}{9a_p} - \frac{(a_p\alpha_p - c_{1,p})^2}{4a_p}.
\end{aligned}$$

Since $c_{2,p} \leq c_{1,p}$, we have

$$\begin{aligned}
\frac{2(a_p\alpha_p - 2c_{1,p} + c_{2,p})^2}{9a_p} - \frac{(a_p\alpha_p - c_{1,p})^2}{4a_p} & \leq \frac{2(a_p\alpha_p - c_{1,p})^2}{9a_p} - \frac{(a_p\alpha_p - c_{1,p})^2}{4a_p} \\
& = \frac{-(a_p\alpha_p - c_{1,p})^2}{36a_p} < 0.
\end{aligned}$$

Similarly, $\frac{(a_s\alpha_s-2c_{1,s}+c_{2,s})^2}{9a_s} - \frac{(a_s\alpha_s-c_{1,s})^2}{4a_s} + \frac{(\alpha_s-2c_{1,s}+c_{2,s})^2}{9} < 0$. Thus, we have $\Delta_{CG} < \Delta_{CR}$. ∎

Corollary 4.4.6 *If we suppress (i) the savings in the waste treatment cost and (ii) the savings in the variable production costs of both products, and assume that competitors have cost advantages over Firm 1 for both products, then Firm 1 is more willing to implement the symbiotic system when competitors can only produce regular variants as compared to the situation when competitors produce green variants.*

4.4.4 Simultaneously Benefiting the Firm and Its Consumers

It is important to note that our comparison of willingness in Section 4.4.3 across the different settings was purely a relative one. In other words, we did not impose that the value of willingness be non-negative. Clearly, one desirable outcome in support of implementation would be that the willingness be positive. Furthermore, the motivation for implementing the system is higher if consumers (as a whole) benefit from it. In this section, our goal is to identify situations under which the above two conditions are simultaneously satisfied. We need the following additional notation.

Notation:

W_i^b	Consumer welfare before the implementation of symbiotic system under Model i, $i \in \{CR, CG\}$.
W_i^a	Consumer welfare after the implementation of symbiotic system under Model i, $i \in \{CR, CG\}$.
$p_{i,j}^b$	Market price of variant i of Product j before the implementation of symbiotic system, $i \in \{r, g\}$, r (resp., g) represents regular (resp., green); $j \in \{p, s\}$.
$p_{i,j}^a$	Market price of variant i of Product j after the implementation of symbiotic system, $i \in \{r, g\}$, r (resp., g) represents regular (resp., green); $j \in \{p, s\}$.

We first consider competition only for the regular variants (Model CR). The equilibrium prices and production quantities for both the variants of a product (say, Product P) are listed in Table 4.7.

Table 4.7. Equilibrium Results under Competition for Regular Variants, both Before and After the Implementation of the Symbiotic System. Here, q_i^* is the Total Production Quantity of Firm $i; i = 1, 2$.

Model	Before Implementation	After Implementation	
CR	Regular Variant	Regular Variant	Green Variant
p^*	$\frac{(2b_1+1)\alpha+c_1+(2b_1+1)c_2}{4b_1+3}$	$\frac{\alpha+c_1^{'}+c_2}{3}$	$\frac{a\alpha+c_1^{'}}{2}$
$q^* = q_1^* + q_2^*$	$\frac{(2b_1+2)\alpha-c_1-(2b_1+1)c_2}{4b_1+3}$	$\frac{\theta_1(2\alpha-c_1^{'}-c_2)}{3}$	$\frac{(1-\theta_1)(a\alpha-c_1^{'})}{2a}$

Recall from Section 4.2 that consumers' valuation of the regular variant is uniformly distributed between 0 and α with density 1. Therefore, we have

$$\begin{aligned} W_{CR}^b &= \int_{p_{r,p}^b}^{\alpha_p} [1(v_p - p_{r,p}^b)]\,\mathrm{d}v_p + \int_{p_{r,s}^b}^{\alpha_s} [1(v_s - p_{r,s}^b)]\,\mathrm{d}v_s = \frac{(\alpha_p - p_{r,p}^b)^2}{2} + \frac{(\alpha_s - p_{r,s}^b)^2}{2} \\ &= \frac{[(2b_1+2)\alpha_p - c_{1,p} - (2b_1+1)c_{2,p}]^2}{2(4b_1+3)^2} + \frac{[(2b_1+2)\alpha_s - c_{1,s} - (2b_1+1)c_{2,s}]^2}{2(4b_1+3)^2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} W_{CR}^a &= \int_{p_{r,p}^a}^{\alpha_p} [\theta_{1,p}(v_p - p_{r,p}^a)]\,\mathrm{d}v_p + \int_{\frac{p_{g,p}^a}{a_p}}^{\alpha_p} [(1-\theta_{1,p})(a_p v_p - p_{g,p}^a)]\,\mathrm{d}v_p \\ &\quad + \int_{p_{r,s}^a}^{\alpha_s} [\theta_{1,s}(v_s - p_{r,s}^a)]\,\mathrm{d}v_s + \int_{\frac{p_{g,s}^a}{a_s}}^{\alpha_s} [(1-\theta_{1,s})(a_s v_s - p_{g,s}^a)]\,\mathrm{d}v_s \\ &= \frac{\theta_{1,p}(\alpha_p - p_{r,p}^a)^2}{2} + \frac{(1-\theta_{1,p})(a_p\alpha_p - p_{g,p}^a)^2}{2a_p} \\ &\quad + \frac{\theta_{1,s}(\alpha_s - p_{r,s}^a)^2}{2} + \frac{(1-\theta_{1,s})(a_s\alpha_s - p_{g,s}^a)^2}{2a_s} \\ &= \frac{\theta_{1,p}(2\alpha_p - c_{1,p}^{'} - c_{2,p})^2}{18} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c_{1,p}^{'})^2}{8a_p} \\ &\quad + \frac{\theta_{1,s}(2\alpha_s - c_{1,s}^{'} - c_{2,s})^2}{18} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c_{1,s}^{'})^2}{8a_s}. \end{aligned}$$

The following result identifies conditions under which consumer welfare improves after the implementation of the system.

Theorem 4.4.7 *If $a_p \geq 2$, $a_s \geq 2$, $c'_{1,p} \leq c_{1,p}$, and $c'_{1,s} \leq c_{1,s}$, then we have $W^a_{CR} > W^b_{CR}$.*

Proof: Since $\frac{\partial \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{8a_p}}{\partial a_p} = \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})(a_p\alpha_p + c'_{1,p})}{8a_p^2} > 0$, the quantity $\frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{8a_p}$ increases with an increase in a_p. Thus, for $a_p \geq 2$, we have

$$\begin{aligned}\frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{8a_p} &\geq \frac{(1-\theta_{1,p})(2\alpha_p - c'_{1,p})^2}{16} \\ &> \frac{(1-\theta_{1,p})(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\theta_{1,p}(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{8a_p} \\ &> \frac{\theta_{1,p}(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18} + \frac{(1-\theta_{1,p})(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18} = \frac{(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18}.\end{aligned}$$

Next, we show $\frac{[(2b_1+2)\alpha_p - c_{1,p} - (2b_1+1)c_{2,p}]^2}{2(4b_1+3)^2}$ reaches its maximum at $b_1 = 0$.

$$\begin{aligned}&\frac{[2\alpha_p - c_{1,p} - 2c_{2,p}]^2}{18} - \frac{[(2b_1+2)\alpha_p - c_{1,p} - (2b_1+1)c_{2,p}]^2}{2(4b_1+3)^2} \\ &= \frac{1}{18(4b_1+3)^2}\{[(4b_1+3)(2\alpha_p - c_{1,p} - c_{2,p})]^2 - [3(2b_1+2)\alpha_p - 3c_{1,p} - 3(2b_1+1)c_{2,p}]^2\} \\ &= \frac{2b_1(\alpha_p - 2c_{1,p} + c_{2,p})}{18(4b_1+3)^2}[(14b_1+12)\alpha_p - (4b_1+6)c_{1,p} - (10b_1+6)c_{2,p}] \geq 0.\end{aligned}$$

Since $c'_{1,p} \leq c_{1,p}$, we have

$$\frac{[(2b_1+2)\alpha_p - c_{1,p} - (2b_1+1)c_{2,p}]^2}{2(4b_1+3)^2} \leq \frac{(2\alpha_p - c_{1,p} - 2c_{2,p})^2}{18} \leq \frac{(2\alpha_p - c'_{1,p} - 2c_{2,p})^2}{18}.$$

Therefore,

$$\frac{\theta_{1,p}(2\alpha_p - c'_{1,p} - c_{2,p})^2}{18} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{8a_p} > \frac{[(2b_1+2)\alpha_p - c_{1,p} - (2b_1+1)c_{2,p}]^2}{2(4b_1+3)^2}.$$

Similarly,

$$\frac{\theta_{1,s}(2\alpha_s - c'_{1,s} - c_{2,s})^2}{18} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c'_{1,s})^2}{8a_s} > \frac{[(2b_1+2)\alpha_s - c_{1,s} - (2b_1+1)c_{2,s}]^2}{2(4b_1+3)^2}.$$

Thus, we have $W^a_{CR} > W^b_{CR}$. ∎

Corollary 4.4.8 *When Firm 1 competes with firms who only produce regular variants, if the green consumers' appreciation for the green variant is relatively high, and the implementation of the symbiotic system does not increase the variable production costs of both products, then consumer welfare increases after the implementation.*

The result in Theorem 4.4.7 can be used to identify a special case in which both Firm 1 and consumers benefit from the implementation of the symbiotic system.

Theorem 4.4.9 *If* $a_p \geq 2$, $a_s \geq 2$, $c'_{1,p} \leq c_{1,p}$, $c'_{1,s} \leq c_{1,s}$, *and* $K \leq \frac{(1-\theta_{1,p})}{8}[(2\alpha_p - c_{1,p})^2 - (\alpha_p - 2c_{1,p} + c_{2,p})^2] + \frac{(1-\theta_{1,s})}{8}[(2\alpha_s - c_{1,s})^2 - (\alpha_s - 2c_{1,s} + c_{2,s})^2]$, *then we have* $W^a_{CR} > W^b_{CR}$ *and* $\Delta_{CR} > 0$.

Proof: Since $a_p \geq 2$, $a_s \geq 2$, $c'_{1,p} \leq c_{1,p}$, and $c'_{1,s} \leq c_{1,s}$, we have $W^a_{CR} > W^b_{CR}$ from Theorem 4.4.7.

By using an approach similar to that in the proof of Theorem 12, we can show the following two properties:

(i) $\frac{(b_1+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1+3)^2}$ reaches its maximum when $b_1 = 0$,

(ii) $\frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{4a_p}$ increases with an increase in a_p.

Thus, for $a_p \geq 2$, we have

$$
\begin{aligned}
\frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{4a_p} &\geq \frac{(1-\theta_{1,p})(2\alpha_p - c_{1,p})^2}{8}, \\
\frac{(b_1+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1+3)^2} &\leq \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \frac{\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{4a_p} - \frac{(b_1+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1+3)^2} \\
\geq \; & \frac{\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(2\alpha_p - c_{1,p})^2}{8} - \frac{(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} \\
= \; & \frac{(1-\theta_{1,p})(2\alpha_p - c_{1,p})^2}{8} - \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} \\
\geq \; & \frac{(1-\theta_{1,p})(2\alpha_p - c_{1,p})^2}{8} - \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{8} \\
= \; & \frac{(1-\theta_{1,p})}{8}[(2\alpha_p - c_{1,p})^2 - (\alpha_p - 2c_{1,p} + c_{2,p})^2] \\
= \; & \frac{(1-\theta_{1,p})}{8}[(\alpha_p + c_{1,p} - c_{2,p})(3\alpha_p - 3c_{1,p} + c_{2,p})] > 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \frac{\theta_{1,s}(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c'_{1,s})^2}{4a_s} - \frac{(b_1+1)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{(4b_1+3)^2} \\
\geq \; & \frac{(1-\theta_{1,s})}{8}[(2\alpha_s - c_{1,s})^2 - (\alpha_s - 2c_{1,s} + c_{2,s})^2].
\end{aligned}
$$

Thus, if $K \leq \frac{(1-\theta_{1,p})}{8}[(2\alpha_p - c_{1,p})^2 - (\alpha_p - 2c_{1,p} + c_{2,p})^2] + \frac{(1-\theta_{1,s})}{8}[(2\alpha_s - c_{1,s})^2 - (\alpha_s - 2c_{1,s} + c_{2,s})^2]$, then we have $\Delta_{CR} > 0$. The result follows. ■

Corollary 4.4.10 *If (i) green consumers' appreciation of the green variant is relatively high, (ii) the implementation of the symbiotic system reduces the variable production costs of both products, and (iii) the fixed cost of implementation of the symbiotic system is modest, then the willingness of Firm 1 to implement the symbiotic system under competition for the regular variants is positive. Also, consumer welfare increases after the implementation.*

To extend the line of thought in Theorem 4.4.9, note that a further motivation for implementation occurs when the willingness of Firm 1 in a monopoly is negative. The following result illustrates one such situation.

Theorem 4.4.11 *If* $a_p = 2$, $a_s = 2$, $c_{2,p} \leq c_{1,p} = c'_{1,p}$, $c_{2,s} \leq c_{1,s} = c'_{1,s}$, $b_1 = 0$, *and* $K = \frac{(1-\theta_{1,p})}{8}[(2\alpha_p - c_{1,p})^2 - (\alpha_p - 2c_{1,p} + c_{2,p})^2] + \frac{(1-\theta_{1,s})}{8}[(2\alpha_s - c_{1,s})^2 - (\alpha_s - 2c_{1,s} + c_{2,s})^2]$, *then we have* $W^a_{CR} > W^b_{CR}$, $\Delta_M < 0$, *and* $\Delta_{CR} > 0$.

Proof: From Theorem 4.4.9, we have $W^a_{CR} > W^b_{CR}$ and $\Delta_{CR} > 0$. Since $a_p = 2$, $a_s = 2$, $c'_{1,p} = c_{1,p}$, $c'_{1,s} = c_{1,s}$, $b_1 = 0$, we have

$$\begin{aligned}
\Delta_M &= \frac{\theta_{1,p}(\alpha_p - c'_{1,p})^2}{4} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c'_{1,p})^2}{4a_p} - \frac{(\alpha_p - c_{1,p})^2}{4(1+b_1)} \\
&\quad + \frac{\theta_{1,s}(\alpha_s - c'_{1,s})^2}{4} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c'_{1,s})^2}{4a_s} - \frac{(\alpha_s - c_{1,s})^2}{4(1+b_1)} - K \\
&= \frac{\theta_{1,p}(\alpha_p - c_{1,p})^2}{4} + \frac{(1-\theta_{1,p})(2\alpha_p - c_{1,p})^2}{8} - \frac{(\alpha_p - c_{1,p})^2}{4} \\
&\quad + \frac{\theta_{1,s}(\alpha_s - c_{1,s})^2}{4} + \frac{(1-\theta_{1,s})(2\alpha_s - c_{1,s})^2}{8} - \frac{(\alpha_s - c_{1,s})^2}{4} - K \\
&= \frac{(1-\theta_{1,p})[(2\alpha_p - c_{1,p})^2 - 2(\alpha_p - c_{1,p})^2]}{8} + \frac{(1-\theta_{1,s})[(2\alpha_s - c_{1,s})^2 - 2(\alpha_s - 2c_{1,s})^2]}{8} - K.
\end{aligned}$$

Using $K = \frac{(1-\theta_{1,p})}{8}[(2\alpha_p - c_{1,p})^2 - (\alpha_p - 2c_{1,p} + c_{2,p})^2] + \frac{(1-\theta_{1,s})}{8}[(2\alpha_s - c_{1,s})^2 - (\alpha_s - 2c_{1,s} + c_{2,s})^2]$, we get $\Delta_M = \frac{(1-\theta_{1,p})}{8}[(\alpha_p - 2c_{1,p} + c_{2,p})^2 - 2(\alpha_p - c_{1,p})^2] + \frac{(1-\theta_{1,s})}{8}[(\alpha_s - 2c_{1,s} + c_{2,s})^2 - 2(\alpha_s - 2c_{1,s})^2]$.

Also, $c_{2,p} \leq c_{1,p}$, $c_{2,s} \leq c_{1,s}$. Therefore,

$$\begin{aligned}
\Delta_M &= \frac{(1-\theta_{1,p})}{8}[(\alpha_p - 2c_{1,p} + c_{2,p})^2 - 2(\alpha_p - c_{1,p})^2] \\
&\quad + \frac{(1-\theta_{1,s})}{8}[(\alpha_s - 2c_{1,s} + c_{2,s})^2 - 2(\alpha_s - 2c_{1,s})^2] \\
&< \frac{(1-\theta_{1,p})}{8}[(\alpha_p - 2c_{1,p} + c_{2,p})^2 - (\alpha_p - c_{1,p})^2] \\
&\quad + \frac{(1-\theta_{1,s})}{8}[(\alpha_s - 2c_{1,s} + c_{2,s})^2 - (\alpha_s - 2c_{1,s})^2] \\
&= \frac{(1-\theta_{1,p})(c_{2,p} - c_{1,p})}{8}[(\alpha_p - 2c_{1,p} + c_{2,p}) + (\alpha_p - c_{1,p})] \\
&\quad + \frac{(1-\theta_{1,s})(c_{2,s} - c_{1,s})}{8}[(\alpha_s - 2c_{1,s} + c_{2,s}) + (\alpha_s - 2c_{1,s})] \leq 0.
\end{aligned}$$

The result follows. ∎

Corollary 4.4.12 *If (1) green consumers appreciate the green variant about two times the regular variant, (2) implementation of the symbiotic system does not change the variable production costs of both products, (3) competitors have a cost advantage over Firm 1, (4) savings in the waste treatment cost are negligible, and (5) the fixed cost of implementation of the symbiotic system is modest, then (i) willingness of Firm 1 to implement is negative in a monopoly, but positive under competition for the regular variants and (ii) consumer welfare increases after the implementation.*

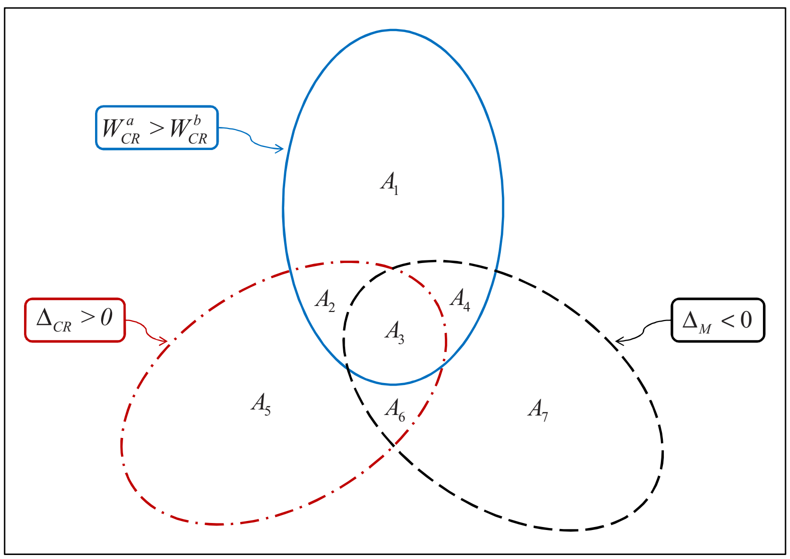


Figure 4.7. A Pictorial Representation of the Conditions in Theorems 4.4.7, 4.4.9, and 4.4.11.

The nested relationships among the results in Theorems 4.4.7, 4.4.9, and 4.4.11, are illustrated in Figure 4.7. The conditions under which $W_{CR}^a > W_{CR}^b$ are represented by the region $A_1 \cup A_2 \cup A_3 \cup A_4$; one such condition forms the premise of Theorem 4.4.7. Similarly, the conditions under which $\Delta_{CR} > 0$ (resp., $\Delta_M < 0$) are represented by the region $A_2 \cup A_3 \cup A_5 \cup A_6$ (resp., $A_3 \cup A_4 \cup A_6 \cup A_7$). Thus, the condition assumed in Theorem 4.4.9 belongs to $A_2 \cup A_3$, while that in Theorem 4.4.11 belongs to Region A_3.

Next, consider competition for both regular and green variants (Model CG). The equilibrium prices and production quantities for both the variants of a product (say, Product P) are listed in Table 4.8.

Table 4.8. Equilibrium Results under Competition for both Regular and Green Variants, both before and after the Implementation of the Symbiotic System.

Model CG	Before Implementation		After Implementation	
	Regular Variant	Green Variant	Regular Variant	Green Variant
p^*	$\frac{(2b_1\theta_1+1)\alpha+c_1+(2b_1\theta_1+1)c_2}{3+4b_1\theta_1}$	$\frac{a\alpha+c_2}{2}$	$\frac{\alpha+c_1^{'}+c_2}{3}$	$\frac{a\alpha+c_1^{'}+c_2}{3}$
$q_1^*+q_2^*$	$\frac{\theta_1[(2b_1\theta_1+2)\alpha-c_1-(2b_1\theta_1+1)c_2]}{3+4b_1\theta_1}$	$\frac{(1-\theta_1)(a\alpha-c_2)}{2a}$	$\frac{\theta_1(2\alpha-c_1^{'}-c_2)}{3}$	$\frac{(1-\theta_1)(2a\alpha-c_1^{'}-c_2)}{3a}$

We first derive consumer surplus before and after the implementation of a symbiotic system.

$$\begin{aligned}
W_{CG}^b &= \frac{\theta_{1,p}(\alpha_p-p_{r,p}^b)^2}{2}+\frac{(1-\theta_{1,p})(a_p\alpha_p-p_{g,p}^b)^2}{2a_p} \\
&\quad+\frac{\theta_{1,s}(\alpha_s-p_{r,s}^b)^2}{2}+\frac{(1-\theta_{1,s})(a_s\alpha_s-p_{g,s}^b)^2}{2a_s}, \\
W_{CG}^a &= \frac{\theta_{1,p}(\alpha_p-p_{r,p}^a)^2}{2}+\frac{(1-\theta_{1,p})(a_p\alpha_p-p_{g,p}^a)^2}{2a_p} \\
&\quad+\frac{\theta_{1,s}(\alpha_s-p_{r,s}^a)^2}{2}+\frac{(1-\theta_{1,s})(a_s\alpha_s-p_{g,s}^a)^2}{2a_s}.
\end{aligned}$$

As with Model CR, our effort here is to identify conditions that benefit both Firm 1 and consumers. Theorem 4.4.13 is used to establish Theorem 4.4.15, which identifies such conditions.

Theorem 4.4.13 *If $c_{1,p}^{'} \leq c_{1,p}$ and $c_{1,s}^{'} \leq c_{1,s}$, then we have $W_{CG}^a > W_{CG}^b$.*

Proof: The consumer welfare before and after the implementation are:

$$\begin{aligned}
W_{CR}^b &= \frac{\theta_{1,p}(\alpha_p-p_{r,p}^b)^2}{2}+\frac{(1-\theta_{1,p})(a_p\alpha_p-p_{g,p}^b)^2}{2a_p} \\
&\quad+\frac{\theta_{1,s}(\alpha_s-p_{r,s}^b)^2}{2}+\frac{(1-\theta_{1,s})(a_s\alpha_s-p_{g,s}^b)^2}{2a_s}.
\end{aligned}$$

$$\begin{aligned} W_{CR}^a &= \frac{\theta_{1,p}(\alpha_p - p_{r,p}^a)^2}{2} + \frac{(1-\theta_{1,p})(a_p\alpha_p - p_{g,p}^a)^2}{2a_p} \\ &\quad + \frac{\theta_{1,s}(\alpha_s - p_{r,s}^a)^2}{2} + \frac{(1-\theta_{1,s})(a_s\alpha_s - p_{g,s}^a)^2}{2a_s}. \end{aligned}$$

Thus, if we can show $p_{r,p}^b \geq p_{r,p}^a$, $p_{g,p}^b \geq p_{g,p}^a$, $p_{r,s}^b \geq p_{r,s}^a$, and $p_{g,s}^b \geq p_{g,s}^a$, then we have $W_{CG}^a > W_{CG}^b$. We first compare the prices of the regular variants of Product P before and after the implementation.

$$\begin{aligned} p_{r,p}^b - p_{r,p}^a &= \frac{(2b_1\theta_{1,p}+1)\alpha_p + c_{1,p} + (2b_1\theta_{1,p}+1)c_{2,p}}{3+4b_1\theta_{1,p}} - \frac{\alpha_p + c_{1,p}^{'} + c_{2,p}}{3} \\ &= \frac{2b_1\theta_{1,p}\alpha_p + 3c_{1,p} - (3+4b_1\theta_{1,p})c_{1,p}^{'} + 2b_1\theta_{1,p}c_{2,p}}{3(3+4b_1\theta_{1,p})}. \end{aligned}$$

Since $c_{1,p}^{'} \leq c_{1,p}$, we have

$$\begin{aligned} p_{r,p}^b - p_{r,p}^a &\geq \frac{2b_1\theta_{1,p}\alpha_p + 3c_{1,p} - (3+4b_1\theta_{1,p})c_{1,p} + 2b_1\theta_{1,p}c_{2,p}}{3(3+4b_1\theta_{1,p})} \\ &= \frac{2b_1\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})}{3(3+4b_1\theta_{1,p})} \geq 0. \end{aligned}$$

Next, we compare the prices of the green variants of Product P before and after the implementation. $p_{g,p}^b - p_{g,p}^a = \frac{a_p\alpha_p + c_{2,p}}{2} - \frac{a_p\alpha_p + c_{1,p}^{'} + c_{2,p}}{3} = \frac{a_p\alpha_p - 2c_{1,p}^{'} + c_{2,p}}{6} \geq 0.$

Similarly, for Product S, we have $p_{r,s}^b \geq p_{r,s}^a$ and $p_{g,s}^b \geq p_{g,s}^a$. Thus, $W_{CG}^a > W_{CG}^b$. ■

Corollary 4.4.14 *When competitors produce both regular and green variants, if the implementation of the symbiotic system reduces the variable production costs of both products, then consumer welfare increases after the implementation.*

Theorem 4.4.15 *If* $K \leq \frac{(1-\theta_{1,p})(a_p\alpha_p - 2c_{1,p}^{'} + c_{2,p})^2}{9a_p} + \frac{(1-\theta_{1,s})(a_s\alpha_s - 2c_{1,s}^{'} + c_{2,s})^2}{9a_s}$, $c_{1,p}^{'} \leq c_{1,p}$, *and* $c_{1,s}^{'} \leq c_{1,s}$, *then we have* $W_{CG}^a > W_{CG}^b$ *and* $\Delta_{CG} > 0$.

Proof: From Theorem 4.4.13, we have $W^a_{CG} > W^b_{CG}$. Using an approach similar to that in the proof of Theorem 4.4.9, we can establish the following two properties:

(i) $\frac{\theta_{1,p}(b_1\theta_{1,p}+1)(\alpha_p-2c_{1,p}+c_{2,p})^2}{(4b_1\theta_{1,p}+3)^2}$ reaches its maximum at $b_1 = 0$,

(ii) $\frac{(1-\theta_{1,p})(a_p\alpha_p-2c'_{1,p}+c_{2,p})^2}{9a_p}$ increases with an increase in a_p.

Then, for $a_p \geq 1$, $c'_{1,p} \leq c_{1,p}$, we have

$$\begin{aligned}
& \frac{\theta_{1,p}(\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(a_p\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9a_p} \\
& -\frac{\theta_{1,p}(b_1\theta_{1,p}+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1\theta_{1,p}+3)^2} \\
\geq \; & \frac{\theta_{1,p}(\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9} - \frac{\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} \\
\geq \; & \frac{\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} - \frac{\theta_{1,p}(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} \\
= \; & \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} > 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{\theta_{1,s}(\alpha_s - 2c'_{1,s} + c_{2,s})^2}{9} + \frac{(1-\theta_{1,s})(a_s\alpha_s - 2c'_{1,s} + c_{2,s})^2}{9a_s} \\
& -\frac{\theta_{1,s}(b_1\theta_{1,s}+1)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{(4b_1\theta_{1,s}+3)^2} \geq \frac{(1-\theta_{1,s})(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta_{CG} \; = \; & \frac{\theta_{1,p}(\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,p})(a_p\alpha_p - 2c'_{1,p} + c_{2,p})^2}{9a_p} \\
& -\frac{\theta_{1,p}(b_1\theta_{1,p}+1)(\alpha_p - 2c_{1,p} + c_{2,p})^2}{(4b_1\theta_{1,p}+3)^2} + \frac{\theta_{1,s}(\alpha_s - 2c'_{1,s} + c_{2,s})^2}{9} \\
& +\frac{(1-\theta_{1,s})(a_s\alpha_s - 2c'_{1,s} + c_{2,s})^2}{9a_s} - \frac{\theta_{1,s}(b_1\theta_{1,s}+1)(\alpha_s - 2c_{1,s} + c_{2,s})^2}{(4b_1\theta_{1,s}+3)^2} - K \\
\geq \; & \frac{(1-\theta_{1,p})(\alpha_p - 2c_{1,p} + c_{2,p})^2}{9} + \frac{(1-\theta_{1,s})(\alpha_s - 2c_{1,s} + c_{2,s})^2}{9} \\
& -K.
\end{aligned}$$

Therefore, if $K \leq \frac{(1-\theta_{1,p})(\alpha_p-2c_{1,p}+c_{2,p})^2}{9} + \frac{(1-\theta_{1,s})(\alpha_s-2c_{1,s}+c_{2,s})^2}{9}$, then we have $\Delta_{CG} > 0$. The result follows. ■

Corollary 4.4.16 *If (i) the implementation of the symbiotic system reduces the variable production costs of both products and (ii) the fixed cost of implementation of the symbiotic system is modest, then the willingness of Firm 1 to implement the symbiotic system under competition for both regular and green variants is positive. Also, consumer welfare increases after the implementation.*

The following result extends Theorem 4.4.15 by further requiring that the willingness of implementation in a monopoly be negative.

Theorem 4.4.17 *If $a_p = 1$, $a_s = 1$, $c^{'}_{1,p} = c_{1,p}$, $c^{'}_{1,s} = c_{1,s}$, $b_1 = 0$, and $\frac{(1-\theta_{1,p})(\alpha_p-2c_{1,p}+c_{2,p})^2}{9} + \frac{(1-\theta_{1,s})(\alpha_s-2c_{1,s}+c_{2,s})^2}{9} \geq K > 0$, then we have $W^a_{CG} > W^b_{CG}$ and $\Delta_{CG} > 0 > \Delta_M$.*

Proof: Since $c^{'}_{1,p} \leq c_{1,p}$, $c^{'}_{1,s} \leq c_{1,s}$, and $K \leq \frac{(1-\theta_{1,p})(\alpha_p-2c_{1,p}+c_{2,p})^2}{9} + \frac{(1-\theta_{1,s})(\alpha_s-2c_{1,s}+c_{2,s})^2}{9}$, then from Theorem 4.4.15, we have $W^a_{CG} > W^b_{CG}$ and $\Delta_{CG} > 0$. If $a_p = 1$, $a_s = 1$, $c^{'}_{1,p} = c_{1,p}$, $c^{'}_{1,s} = c_{1,s}$, $b_1 = 0$, we have

$$
\begin{aligned}
\Delta_M &= \frac{\theta_{1,p}(\alpha_p - c^{'}_{1,p})^2}{4} + \frac{(1-\theta_{1,p})(a_p\alpha_p - c^{'}_{1,p})^2}{4a_p} - \frac{(\alpha_p - c_{1,p})^2}{4(1+b_1)} \\
&\quad + \frac{\theta_{1,s}(\alpha_s - c^{'}_{1,s})^2}{4} + \frac{(1-\theta_{1,s})(a_s\alpha_s - c^{'}_{1,s})^2}{4a_s} - \frac{(\alpha_s - c_{1,s})^2}{4(1+b_1)} - K \\
&= \frac{\theta_{1,p}(\alpha_p - c_{1,p})^2}{4} + \frac{(1-\theta_{1,p})(\alpha_p - c_{1,p})^2}{4} - \frac{(\alpha_p - c_{1,p})^2}{4} \\
&\quad + \frac{\theta_{1,s}(\alpha_s - c_{1,s})^2}{4} + \frac{(1-\theta_{1,s})(\alpha_s - c_{1,s})^2}{4} - \frac{(\alpha_s - c_{1,s})^2}{4} - K \\
&= -K.
\end{aligned}
$$

Thus, for any $K > 0$, we have $\Delta_M < 0$. The result follows. ■

Corollary 4.4.18 *If (i) the green consumers' appreciation for the green variant is negligible, (ii) the implementation of the symbiotic system does not change the variable production costs of both products, and (iii) the savings in the waste treatment costs are negligible, then Firm 1 is more willing to implement the symbiotic system under the competition with firms who produce both regular and green variants. The consumer welfare also increases after the implementation.*

To summarize, Theorems 4.4.7–4.4.17 and Corollaries 4.4.8–4.4.18 together, suggest that competition for either the regular variant or both regular and green variants can indeed shift a firm's willingness from negative to positive, and improve consumer welfare as well. This perhaps explains the timing of SPB's decision to implement the paper-sugar complex (see Section 4.1.1).

4.5 Directions for Future Work

Going back to the real-world implementation described in Section 4.1.1, the unique symbiotic model followed by SPB has transformed the livelihood of the local farming community and has provided a reliable supply of raw material for both SPB and Ponni Sugars. The rain shadow region around the paper-sugar complex has witnessed a green revolution, through the irrigation of more than 1500 acres of dry and fallow land with treated effluent. For the government, the positive societal impact of this implementation provides an ideal case study to identify and incentivize similar symbiotic systems. Some examples of possible incentives structures that can be studied include (1) providing a one-time subsidy, (2) providing long-term tax relief, (3) subsidizing the supply of electricity and water, and (4) improving societal consciousness of symbiotic initiatives. To cite an instance, the recycling facilities established through the Eco-Town Program in Kawasakia, Japan, received an average investment subsidy of 48% from the government (Van Berkel et al. 2009).

In the models analyzed in this paper, we considered two products that are symbiotically connected. A much-larger example of Burnside Industrial Park, which involves about 1300 businesses, is described in Noronha (1999). The businesses within the industrial park are involved in (1) scavenger roles (reuse, remanufacture, refurbish, repair and recover), (2) decomposer roles (recycling), (3) producing/selling environment-friendly products, and (4) providing environmental management services. These companies deal in a diverse range of materials and have established complex relationships within the park. This example motivates the notion of *Design for Symbiosis*: problems of designing the layout of a symbiotic system, coordinating the material exchanges between the firms, and scheduling the logistics activities.

For the two-product symbiotic system analyzed in this paper, we assumed that the production decisions of both the products are taken by a common firm interested in maximizing its total profit. Under this centralized setting, even if the implementation of the system hurts the profit from one product, the firm may be willing to implement if total profit improves. Once the system is implemented, the complete exchange of wastes between the two production processes is guaranteed. However, if the products are manufactured by two independent firms, then participation in a symbiotic relationship becomes voluntary. Thus, in a decentralized setting, the problem of pricing the wastes becomes relevant. On the one hand, if the price of one type of waste is too low, then the firm that generates the waste may be less willing to supply it to the other firm (that uses this waste as a raw material). On the other hand, if this price is excessive, then the firm that can potentially purchase the waste may switch to an alternative source of raw material. Thus, the pricing of the wastes affects not only the revenues of the firms involved but also their operations.

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